DYNAMICAL BIFURCATIONS OF LOW-DIMENSIONAL SYSTEMS

Scope:

- To show how to apply the MSM to analyze multiple bifurcations of simple low-dimensional systems;
- To show general results, valid also for systems of arbitrary dimensions.

Outline:

1. Simple-Hopf bifurcation (codimension-1).
2. Non-resonant Double-Hopf bifurcation (codimension-2).
3. Divergence-Hopf bifurcation (codimension-2)
4. 1:1 (not generic) and 1:3 resonant double-Hopf bifurcations (codimension-3)
5. 1:2 resonant double-Hopf bifurcations (codimension-3)
6. Defective bifurcation: the double-zero bifurcation (codimension-2)
7. Defective bifurcation: the 1:1 (generic) resonant double-Hopf bifurcation (codimension-3).

1. SIMPLE-HOPF BIFURCATION

EXAMPLE: TWO RAYLEIGH-DUFFING OSCILLATORS, ONE STABLE, THE OTHER UNSTABLE

$$\begin{cases} \ddot{x} - \mu\dot{x} + \omega_1^2 x + b_1\dot{x}^3 + cx^3 - b_0(\dot{y} - \dot{x})^3 = 0 \\ \ddot{y} + \xi\dot{y} + \omega_2^2 y + b_2\dot{y}^3 + cy^3 + b_0(\dot{y} - \dot{x})^3 = 0 \end{cases}$$

with $\xi > 0$.

- Discussion on stability:

When $\mu < 0$ the origin is stable. When μ crosses the bifurcation value $\mu_c = 0$, the origin becomes unstable. The system undergoes a Hopf bifurcation. Although the *y*-oscillator is stable, it contributes to the motion.

- Rescaling:

$$\mu \to \varepsilon\mu \quad , \quad (x, y) \to (\varepsilon^{1/2} x, \varepsilon^{1/2} y)$$

- Series expansions:

$$\begin{pmatrix} x(t;\varepsilon) \\ y(t;\varepsilon) \end{pmatrix} = \begin{pmatrix} x_0(t_0, t_1, t_2, \cdots) \\ y_0(t_0, t_1, t_2, \cdots) \end{pmatrix} + \varepsilon \begin{pmatrix} x_1(t_0, t_1, t_2, \cdots) \\ y_1(t_0, t_1, t_2, \cdots) \end{pmatrix} + \varepsilon^2 \begin{pmatrix} x_2(t_0, t_1, t_2, \cdots) \\ y_2(t_0, t_1, t_2, \cdots) \end{pmatrix} + \cdots$$

where $t_k \coloneqq \varepsilon^k t_k$.

- Chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}t} = \mathrm{d}_0 + \varepsilon\, \mathrm{d}_1 + \varepsilon^2\, \mathrm{d}_2 + \cdots, \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2} = \mathrm{d}_0^2 + 2\varepsilon\, \mathrm{d}_0\, \mathrm{d}_1 + \varepsilon^2 \left(\mathrm{d}_1^2 + 2\, \mathrm{d}_0\, \mathrm{d}_2 \right) + \cdots$$

where $\mathrm{d}_k \coloneqq \partial / \partial t_k$.

- Perturbation equations:

$$
\varepsilon^0 : \begin{cases} \mathrm{d}_0^2\, x_0 + \omega_1^2 x_0 = 0 \\ \mathrm{d}_0^2\, y_0 + \xi\, \mathrm{d}_0 y_0 + \omega_2^2 y_0 = 0 \end{cases}
$$

$$
\varepsilon^1 : \begin{cases} \mathrm{d}_0^2\, x_1 + \omega_1^2 x_1 = -2\,\mathrm{d}_0 \mathrm{d}_1 x_0 - b_1 (\mathrm{d}_0 x_0)^3 - c x_0^3 + b_0 (\mathrm{d}_0 y_0 - \mathrm{d}_0 x_0)^3 + \mu\, \mathrm{d}_0 x_0 \\ \mathrm{d}_0^2\, y_1 + \xi\, \mathrm{d}_0 y_1 + \omega_2^2 y_1 = -2\,\mathrm{d}_0 \mathrm{d}_1 y_0 - b_2 (\mathrm{d}_0 y_0)^3 - c y_0^3 - b_0 (\mathrm{d}_0 y_0 - \mathrm{d}_0 x_0)^3 - \xi\, \mathrm{d}_1 y_0 \end{cases}
$$

$$
\varepsilon^2 : \begin{cases} \mathrm{d}_0^2\, x_2 + \omega_1^2 x_2 = -(2\,\mathrm{d}_0 \mathrm{d}_2 x_0 + \mathrm{d}_1^2 x_0 + 2\,\mathrm{d}_0 \mathrm{d}_1 x_1) - 3 b_1 (\mathrm{d}_0 x_0)^2 (\mathrm{d}_1 x_0 + \mathrm{d}_0 x_1) \\ \qquad -3 c x_0^2 x_1 + 3 b_0 (\mathrm{d}_0 y_0 - \mathrm{d}_0 x_0)^2 (\mathrm{d}_0 y_1 + \mathrm{d}_1 y_0 - \mathrm{d}_0 x_1 - \mathrm{d}_1 x_0) \\ \qquad +\mu (\mathrm{d}_1 x_0 + \mathrm{d}_0 x_1) \\ \mathrm{d}_0^2\, y_2 + \xi\, \mathrm{d}_0 y_2 + \omega_2^2 y_2 = -(2\,\mathrm{d}_0 \mathrm{d}_2 y_0 + \mathrm{d}_1^2 y_0 + 2\,\mathrm{d}_0 \mathrm{d}_1 y_1) - 3 b_2 (\mathrm{d}_0 y_0)^2 (\mathrm{d}_1 y_0 + \mathrm{d}_0 y_1) \\ \qquad -3 c y_0^2 y_1 - 3 b_0 (\mathrm{d}_0 y_0 - \mathrm{d}_0 x_0)^2 (\mathrm{d}_0 y_1 + \mathrm{d}_1 y_0 - \mathrm{d}_0 x_1 - \mathrm{d}_1 x_0) \\ \qquad -\xi (\mathrm{d}_2 y_0 + \mathrm{d}_1 y_1) \end{cases}
$$

- Generating solution:

$$x_0 = A(t_1, t_2)\,\mathrm{e}^{i\omega_1 t_0} + c.c., \quad y_0 = 0$$

since the *y*-oscillator is damped. Therefore: *x* = *active coordinate*, and *y*= *passive coordinate*.

- ε -order:

  ➢ equations:

$$\begin{cases} \mathrm{d}_0^2\, x_1 + \omega_1^2 x_1 = f_{1,1}\,\mathrm{e}^{i\omega_1 t_0} + f_{1,3}\,\mathrm{e}^{3i\omega_1 t_0} + c.c. \\ \mathrm{d}_0^2\, y_1 + \xi\,\mathrm{d}_0 y_1 + \omega_2^2 y_1 = f_{2,1}\,\mathrm{e}^{i\omega_1 t_0} + f_{2,3}\,\mathrm{e}^{3i\omega_1 t_0} + c.c. \end{cases}$$

where:

$$f_{1,1} := -2i\omega_1\,\mathrm{d}_1 A + i\mu\omega_1 A - 3[c + i(b_0 + b_1)\omega_1^3]A^2\bar{A},$$

$$f_{1,3} := [-c + i(b_0 + b_1)\omega_1^3]A^3$$

$$f_{2,1} := 3ib_0\omega_1^3 A^2\bar{A}, \quad f_{2,3} := -ib_0\omega_1^3 A^3$$

- elimination of resonant terms requires $f_{1,1} = 0$, from which:

$$\mathrm{d}_1 A = \frac{1}{2}\mu A + \frac{3}{2}[i\frac{c}{\omega_1} - (b_0 + b_1)\omega_1^2]A^2\bar{A}$$

The *y*-equation, does not require any additional condition.

- Solution:

$$x_1 = -\frac{f_{1,3}}{8\omega_1^2}\mathrm{e}^{3i\omega_1 t_0} + c.c.,$$

$$y_1 = \frac{f_{2,1}}{\omega_2^2 - \omega_1^2 + i\xi\omega_1}\mathrm{e}^{i\omega_1 t_0} + \frac{f_{2,3}}{\omega_2^2 - 9\omega_1^2 + 3i\xi\omega_1}\mathrm{e}^{3i\omega_1 t_0} + c.c.$$

- **Note:** *Only the particular solutions have been considered*, since the complementary *x*-solution repeats the generating one, and the complementary *y*-solution decays in time.

- ε^2-order:

  - equations:

$$\mathrm{d}_0^2 x_2 + \omega_1^2 x_2 = -(2\,\mathrm{d}_0\mathrm{d}_2 x_0 + \mathrm{d}_1^2 x_0 + 2\,\mathrm{d}_0\mathrm{d}_1 x_1) - 3b_1(\mathrm{d}_0 x_0)^2(\mathrm{d}_1 x_0 + \mathrm{d}_0 x_1)$$
$$-3c x_0^2 x_1 + 3b_0(\mathrm{d}_0 x_0)^2(\mathrm{d}_0 y_1 - \mathrm{d}_0 x_1 - \mathrm{d}_1 x_0) + \mu(\mathrm{d}_1 x_0 + \mathrm{d}_0 x_1)$$

The $\mathrm{d}_1^2 x_0$ term requires evaluation of :

$$\mathrm{d}_1^2 A = \frac{1}{2}\mu\,\mathrm{d}_1 A + \frac{3}{2}[i\frac{c}{\omega_1} - (b_0 + b_1)\omega_1^2](2A\overline{A}\,\mathrm{d}_1 A + A^2\,\mathrm{d}_1 \overline{A})$$
$$= \frac{1}{4}\mu^2 A - 3\mu(b_0 + b_1)\omega_1^2 A^2\overline{A}$$
$$-\frac{9}{4}\frac{c^2}{\omega_1^2} - 9i(b_0 + b_1)c\omega_1 + \frac{27}{4}(b_0 + b_1)^2\omega_1^4 A^3\overline{A}^2$$

- elimination of secular terms:

$$\mathrm{d}_2 A = -i\frac{\mu^2}{8\omega_1}A - \frac{3}{4}\frac{c}{\omega_1^2}\mu A^2\bar{A}$$

$$+[-\frac{3}{2}c(b_0+b_1) - i\frac{15}{16}\frac{c^2}{\omega_1^3} + \frac{9}{16}i\omega_1^3(b_0+b_1)^2$$

$$+9ib_0^2\omega_1^5(\frac{1}{2}\frac{1}{\omega_2^2-9\omega_1^2+3i\xi\omega_1}$$

$$+\frac{1}{\omega_2^2-\omega_1^2+i\xi\omega_1} - \frac{1}{2}\frac{1}{\omega_2^2-\omega_1^2-i\xi\omega_1})]A^3\bar{A}^2$$

- Reconstitution method and parameter reabsorbing:

$$\dot{A} = \varepsilon \, \mathrm{d}_1 \, A + \varepsilon^2 \, \mathrm{d}_2 \, A$$

This equation is multiplied by $\varepsilon^{1/2}$ and quantities transformed back as $\varepsilon^{1/2} A \to A, \varepsilon\mu \to \mu$, thus obtaining a *complex bifurcation equation*:

$$\begin{aligned}
\dot{A} = {} & (\frac{1}{2}\mu - i\frac{\mu^2}{8\omega_1})A + \frac{3}{2}[i\frac{c}{\omega_1} - \frac{3}{4}\frac{c}{\omega_1^2}\mu - (b_0 + b_1)\omega_1^2]A^2\bar{A} \\
& + [-\frac{3}{2}c(b_0 + b_1) - i\frac{15}{16}\frac{c^2}{\omega_1^3} + \frac{9}{16}i\omega_1^3(b_0 + b_1)^2 \\
& + 9ib_0^2\omega_1^5(\frac{1}{2}\frac{1}{\omega_2^2 - 9\omega_1^2 + 3i\xi\omega_1} + \frac{1}{\omega_2^2 - \omega_1^2 + i\xi\omega_1} - \frac{1}{2}\frac{1}{\omega_2^2 - \omega_1^2 - i\xi\omega_1})]A^3\bar{A}^2
\end{aligned}$$

Using the polar form:

$$A(t) := \frac{1}{2}a(t)\mathrm{e}^{i\theta(t)}$$

and separating the real and imaginary parts, two real *bifurcation equations* follow.

- Amplitude equation:

$$\dot{a} = \frac{1}{2}\mu a - [\frac{3}{8}(b_0 + b_1)\omega_1^2 - \frac{3}{16}\frac{c}{\omega_1^2}\mu]a^3$$

$$+[-\frac{3}{32}c(b_0 + b_1) + \frac{27}{32}b_0^2\xi\omega_1^6(\frac{1}{(\omega_2^2 - 9\omega_1^2)^2 + 9\xi^2\omega_1^2} + \frac{1}{(\omega_2^2 - \omega_1^2)^2 + \xi^2\omega_1^2})]a^5$$

- phase-equation:

$$
\begin{aligned}
a\dot{\vartheta} = & -\frac{1}{8}\frac{\mu^2}{\omega_1}a + \frac{3}{8}\frac{c}{\omega_1}a^3 \\
& +[\frac{9}{256}(b_0+b_1)^2\omega_1^3 - \frac{15}{256}\frac{c^2}{\omega_1^3} \\
& -\frac{9}{32}b_0^2\omega_1^7(\frac{9}{(\omega_2^2-9\omega_1^2)^2+9\xi^2\omega_1^2}+\frac{1}{(\omega_2^2-\omega_1^2)^2+\xi^2\omega_1^2}) \\
& +\frac{9}{32}b_0^2\omega_1^5\omega_2^2(\frac{1}{(\omega_2^2-9\omega_1^2)^2+9\xi^2\omega_1^2}+\frac{1}{(\omega_2^2-\omega_1^2)^2+\xi^2\omega_1^2})]a^5
\end{aligned}
$$

❑ **Note:** The essential dynamics of the original system is governed by a one-dimensional amplitude-equation.

- Response of the system:

$$x = a(t)\cos(\Phi(t)) + a^3(t)[\frac{1}{32}\frac{c}{\omega_1^2}\cos(3\Phi(t)) + \frac{1}{32}(b_0 + b_1)\omega_1 \sin(3\Phi(t))] + \cdots$$

$$y = \frac{3}{4}a^3(t)[\frac{b_0\xi\omega_1^4}{(\omega_2^2 - \omega_1^2)^2 + \xi^2\omega_1^2}\cos(\Phi(t)) - \frac{b_0\xi\omega_1^4}{(\omega_2^2 - 9\omega_1^2)^2 + 9\xi^2\omega_1^2}\cos(3\Phi(t))$$
$$+ \frac{b_0\omega_1^3(\omega_1^2 - \omega_2^2)}{(\omega_2^2 - \omega_1^2)^2 + \xi^2\omega_1^2}\sin(\Phi(t)) - \frac{b_0\omega_1^3(9\omega_1^2 - \omega_2^2)}{(\omega_2^2 - 9\omega_1^2)^2 + 9\xi^2\omega_1^2}\sin(3\Phi(t))] + \cdots$$

where:

$$\Phi(t) := \omega_1 t + \theta(t)\,.$$

■ Steady solutions:

$$a = a_s = \text{const}\,, \qquad \dot{\theta} =: \kappa = \text{const}$$

are limit cycles, of amplitude a_s and (nonlinear) frequency $\dot{\Phi} = \omega_1 + \kappa = \text{const}$

- Remarks

(a) At leading order, only the active coordinate *x*, contributes to the motion.
(b) At a higher-order, also the passive coordinate *y* is triggered. This *does not contribute with its own free evolution,* but rather *is forced by the active x-coordinate*
(c) Taking into account passive coordinates, *does not increase* the dimension of the amplitude equations, but just gives a more accurate description of the dynamics

2.NON-RESONANT DOUBLE-HOPF BIFURCATION

- When, for a critical combination of the parameters, the generating system admits two or more pairs $\pm i\omega_k \ (k = 1, 2, \cdots)$ of imaginary eigenvalues, a *multiple Hopf bifurcation* takes place.

- More modal components must be considered in the generating solution; the response is said *multi-modal*.

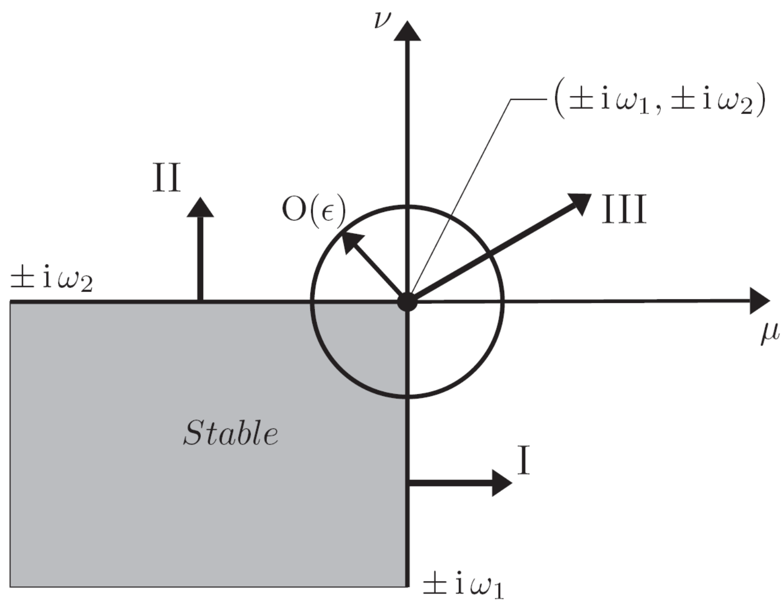
- The response is non-resonant if *none of the active frequencies* ω_k *can be expressed as a linear combination with integer coefficients of the remaining active frequencies*.

EXAMPLE: TWO COUPLED RAYLEIGH-DUFFING OSCILLATORS, BOTH UNSTABLE

$$\begin{cases} \ddot{x} - \mu\dot{x} + \omega_1^2 x + b_1\dot{x}^3 + cx^3 - b_0(\dot{y} - \dot{x})^3 = 0 \\ \ddot{y} - \nu\dot{y} + \omega_2^2 y + b_2\dot{y}^3 + cy^3 + b_0(\dot{y} - \dot{x})^3 = 0 \end{cases}$$

The linear frequencies are incommensurable: $\omega_2 \neq r\omega_1, \forall r \in \mathbb{Q}$

- Linear stability diagram:



Linear stability diagram for a system undergoing double-Hopf bifurcation.

- Rescaling:

$$(\mu,\nu)\to(\varepsilon\mu,\varepsilon\nu)\text{ , }(x,y)\to(\varepsilon^{1/2}x,\varepsilon^{1/2}y)$$

- Series expansions:

$$\begin{pmatrix} x(t;\varepsilon) \\ y(t;\varepsilon) \end{pmatrix} = \begin{pmatrix} x_0(t_0,t_1,t_2,\cdots) \\ y_0(t_0,t_1,t_2,\cdots) \end{pmatrix} + \varepsilon \begin{pmatrix} x_1(t_0,t_1,t_2,\cdots) \\ y_1(t_0,t_1,t_2,\cdots) \end{pmatrix} + \varepsilon^2 \begin{pmatrix} x_2(t_0,t_1,t_2,\cdots) \\ y_2(t_0,t_1,t_2,\cdots) \end{pmatrix} + \cdots$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \mathrm{d}_0 + \varepsilon\,\mathrm{d}_1 + \varepsilon^2\,\mathrm{d}_2 + \cdots, \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2} = \mathrm{d}_0^2 + 2\varepsilon\,\mathrm{d}_0\,\mathrm{d}_1 + \varepsilon^2\left(\mathrm{d}_1^2 + 2\,\mathrm{d}_0\,\mathrm{d}_2\right) + \cdots$$

where $t_k \coloneqq \varepsilon^k t_k$ and $\mathrm{d}_k \coloneqq \partial/\partial t_k$.

- Perturbation equations:

$$\varepsilon^0 : \begin{cases} d_0^2\, x_0 + \omega_1^2 x_0 = 0 \\ d_0^2\, y_0 + \omega_2^2 y_0 = 0 \end{cases}$$

$$\varepsilon^1 : \begin{cases} d_0^2\, x_1 + \omega_1^2 x_1 = -2\, d_0 d_1 x_0 + \mu\, d_0 x_0 - b_1 (d_0 x_0)^3 - c x_0^3 + b_0 (d_0 y_0 - d_0 x_0)^3 \\ d_0^2\, y_1 + \omega_2^2 y_1 = -2\, d_0 d_1 y_0 + \nu\, d_0 y_0 - b_2 (d_0 y_0)^3 - c y_0^3 - b_0 (d_0 y_0 - d_0 x_0)^3 \end{cases}$$

………

- Generating solution:

$$\begin{cases} x_0 = A_1 \left(t_1, t_2, \ldots \right) e^{i\omega_1 t_0} + c.c. \\ y_0 = A_2 \left(t_1, t_2, \ldots \right) e^{i\omega_2 t_0} + c.c. \end{cases}$$

- ε-order:
  - equations:
    Cubic terms produce harmonics $(\omega_1, \omega_2; 3\omega_1, 3\omega_2, \omega_2 \pm 2\omega_1, 2\omega_2 \pm \omega_1)$; among them, only ω_1 in the first equation and ω_2 in the second equation are resonant:

$$\begin{cases} \mathrm{d}_0^2\, x_1 + \omega_1^2 x_1 = f_{1,1}\, \mathrm{e}^{i\omega_1 t_0} + NRT + c.c. \\ \mathrm{d}_0^2\, y_1 + \omega_2^2 y_1 = f_{2,2}\, \mathrm{e}^{i\omega_2 t_0} + NRT + c.c. \end{cases}$$

where:

$$f_{1,1} := -2i\omega_1\, \mathrm{d}_1 A_1 + i\omega_1 \mu A_1 - 3[c + i(b_0 + b_1)\omega_1^3] A_1^2 \overline{A}_1 - 6 b_0 \omega_1 \omega_2^2 A_1 A_2 \overline{A}_2$$
$$f_{2,2} := -2i\omega_2\, \mathrm{d}_1 A_2 + i\nu\omega_2 A_2 - 3[c + i(b_0 + b_2)\omega_2^3] A_2^2 \overline{A}_2 - 6 b_0 \omega_1^2 \omega_2 A_1 \overline{A}_1 A_2$$

➢ Zeroing the secular terms requires $f_{1,1} = f_{2,2} = 0$, from which:

$$\begin{cases} \mathrm{d}_1 A_1 = \frac{1}{2}\mu A_1 + \frac{3}{2}[i\frac{c}{\omega_1} - (b_1 + b_0)\omega_1^2]A_1^2\overline{A}_1 - 3b_0\omega_2^2 A_1 A_2 \overline{A}_2 \\ \mathrm{d}_1 A_2 = \frac{1}{2}\nu A_2 + \frac{3}{2}[i\frac{c}{\omega_2} - (b_2 + b_0)\omega_2^2]A_2^2\overline{A}_2 - 3b_0\omega_1^2 A_1 A_2 \overline{A}_1 \end{cases}$$

- First-order solution:

  The previous equations are multiplied by $\varepsilon^{3/2}$ and use is made of the inverse transformations $\varepsilon^{1/2} A_k \to A_k$, $\varepsilon(\mu, \nu) \to (\mu, \nu)$, $\varepsilon\,\mathrm{d}_1 \to \mathrm{D}$, so that $\mathrm{d}_1 A_k \equiv \dot{A}_k$. By using the polar forms:

$$A_k(t) := \frac{1}{2} a_k(t)\,\mathrm{e}^{i\theta_k(t)} \quad k = 1, 2$$

  four real bifurcation equations follow.

- Amplitude modulation equations:

$$\begin{cases} \dot{a}_1 = \dfrac{1}{2}\mu a_1 - \dfrac{3}{8}(b_0 + b_1)\omega_1^2 a_1^3 - \dfrac{3}{4} b_0 \omega_2^2 a_1 a_2^2 \\ \dot{a}_2 = \dfrac{1}{2}\nu a_2 - \dfrac{3}{8}(b_0 + b_2)\omega_2^2 a_2^3 - \dfrac{3}{4} b_0 \omega_1^2 a_1^2 a_2 \end{cases}$$

- Phase-modulation equations:

$$\begin{cases} a_1\dot{\theta}_1 = \dfrac{3}{8}\dfrac{c}{\omega_1} a_1^3 \\ a_2\dot{\theta}_2 = \dfrac{3}{8}\dfrac{c}{\omega_2} a_2^3 \end{cases}$$

❑ **Note:** i*n the non-resonant case, the real-amplitude equations are uncoupled from the phase-equations* . Therefore, the essential dynamics of the system is governed by the reduced set of two (RAME) equations.

■ Steady solutions, bifurcation chart and bifurcation diagrams

- Steady motions:

Are the fixed points $a_1 = a_{1s} = \text{const}, a_2 = a_{2s} = \text{const}$. Assume $b_1 = b_2 =: b > 0$, $\beta := b_0 / b > 0$. The steady motions are solutions of:

$$\begin{cases} a_1[\dfrac{1}{2}\dfrac{\mu}{b} - \dfrac{3}{8}(1+\beta)\omega_1^2 a_1^2 - \dfrac{3}{4}\beta\omega_2^2 a_2^2] = 0 \\ a_2[\dfrac{1}{2}\dfrac{\nu}{b} - \dfrac{3}{8}(1+\beta)\omega_2^2 a_2^2 - \dfrac{3}{4}\beta\omega_1^2 a_1^2] = 0 \end{cases}$$

- Four essentially different solutions, $(s = T, P_1, P_2, Q)$:

$$(T\ ):\quad a_{1T} = 0,\ a_{2T} = 0,\ \forall(\mu,\nu)$$

$$(P_1\ ):\quad a_{1P} = \frac{1}{\omega_1}\sqrt{\frac{4\mu}{3b(1+\beta)}},\quad a_{2P} = 0,\quad \forall\nu$$

$$(P_2\ ):\quad a_{1P} = 0,\quad a_{2P} = \frac{1}{\omega_2}\sqrt{\frac{4\nu}{3b(1+\beta)}},\quad \forall\mu$$

$$(Q\ ):\quad a_{1Q} = \frac{2}{\omega_1}\sqrt{\frac{2\beta\nu-(1+\beta)\mu}{3b(3\beta^2-2\beta-1)}},\quad a_{2Q} = \frac{2}{\omega_2}\sqrt{\frac{2\beta\mu-(1+\beta)\nu}{3b(3\beta^2-2\beta-1)}}$$

- Meaning of the solutions:

(T) is the *trivial* solution, which corresponds to the equilibrium position of the system.

(P_1) is the mono-modal *periodic* a_1-solution:

$$x = a_{1P}\cos(\Omega_1 t + \theta_{10}), \quad y = 0, \quad \Omega_1 := \omega_1 + \frac{3}{8}\frac{c}{\omega_1}a_{1P}^2$$

(P_2) is the mono-modal *periodic* a_2-solution:

$$x = 0, \quad y = a_{2P}\cos(\Omega_2 t + \theta_{20}), \quad \Omega_2 := \omega_2 + \frac{3}{8}\frac{c}{\omega_2}a_{2P}^2$$

(Q) is a bimodal *quasi-periodic* solution:

$$x = a_{1Q}\cos(\Omega_1 t + \theta_{10}),\; y = a_{2Q}\cos(\Omega_2 t + \theta_{20}),\; \Omega_1 := \omega_1 + \frac{3c}{8\omega_1}a_{1Q}^2,\; \Omega_2 := \omega_2 + \frac{3c}{8\omega_2}a_{2Q}^2$$

since ω_1 and ω_2 are incommensurable.

- Existence domains of the solutions

Since the amplitudes are real and positive:

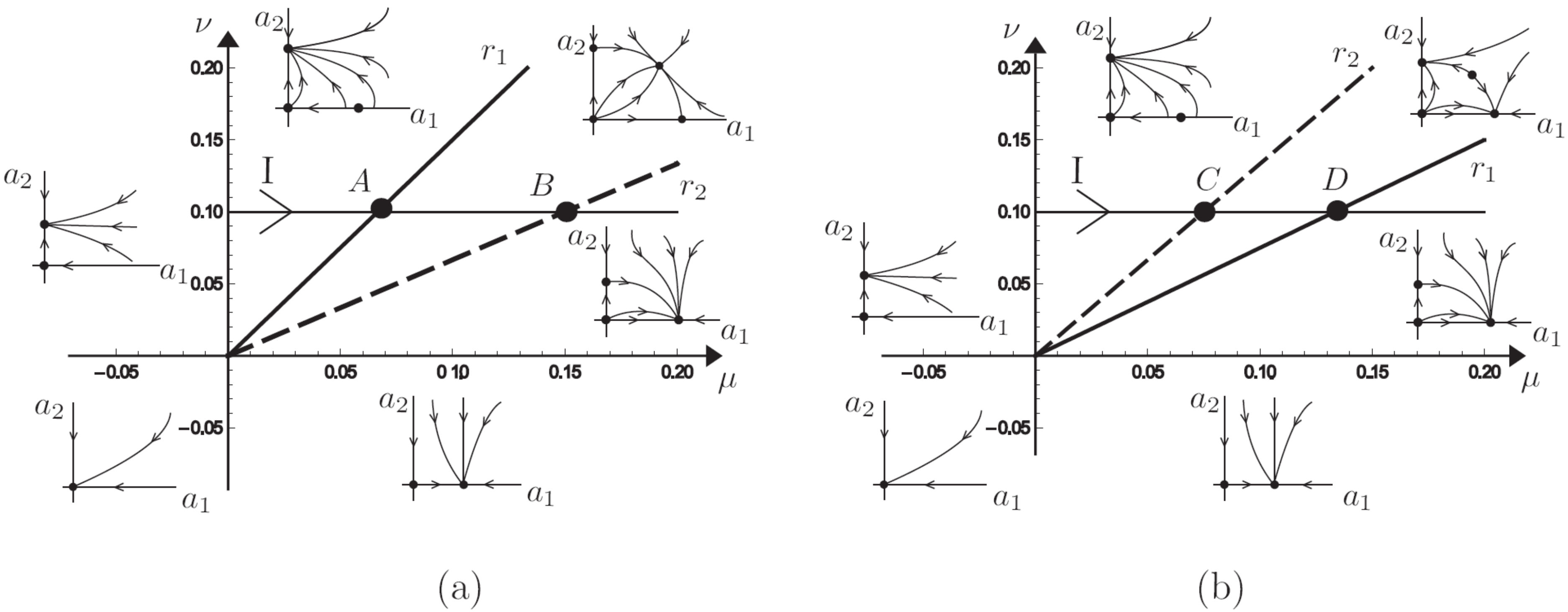
- the T-solution exists in the whole plane
- the P_1-solution is defined in the $\mu \geq 0$ half-plane
- the P_2-solution in the $\nu \geq 0$ half-plane
- the Q-solution requires:

$$\frac{2\beta}{1+\beta}\mu < \nu < \frac{1+\beta}{2\beta}\mu \quad \text{if } \beta < 1$$

$$\frac{1+\beta}{2\beta}\mu < \nu < \frac{2\beta}{1+\beta}\mu \quad \text{if } \beta > 1$$

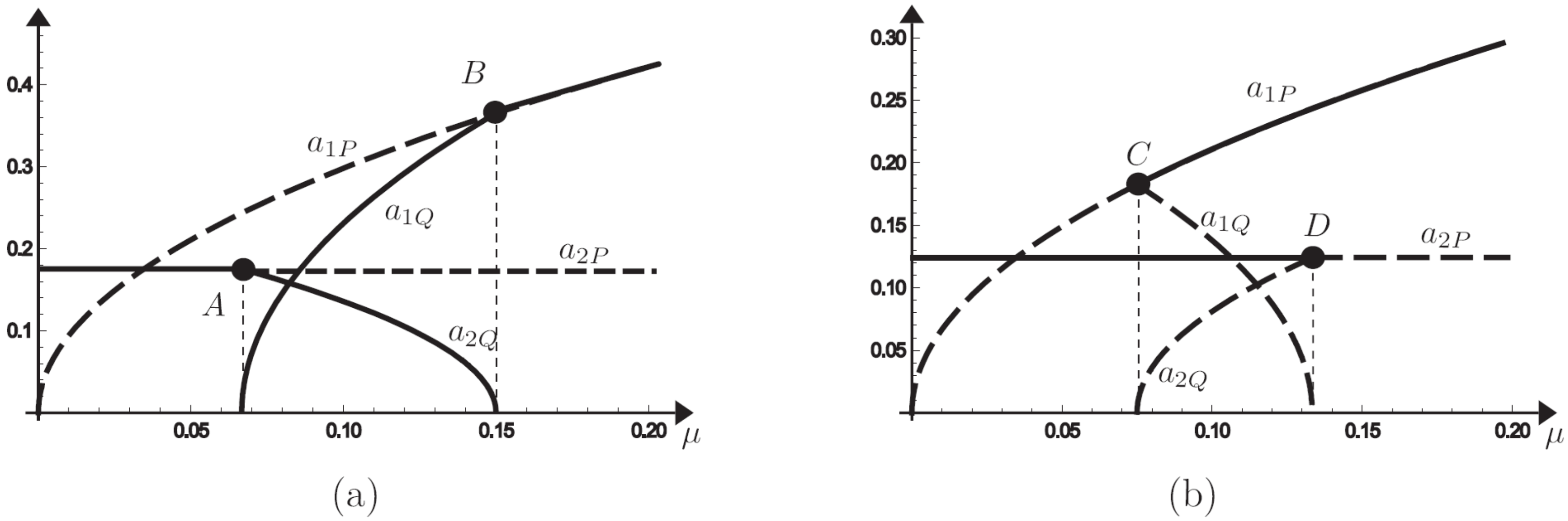
i.e. it exists in the sector bounded by $r_1 := \{(\mu,\nu) \mid \nu = (1+\beta)/(2\beta)\mu\}$, $r_2 := \{(\mu,\nu) \mid \nu = (2\beta)/(1+\beta)\mu\}$.

- At r_1: $a_{1Q}=0$, $a_{2Q}=a_{2P}$; at r_2: $a_{1Q}=a_{1P}$, $a_{2Q}=0$
- r_1 and r_2 are *bifurcation loci*, where a quasi-periodic motion bifurcates from a periodic motion.



Bifurcation chart for: (a) $\beta = 1/2$ and (b) $\beta = 2$

- Planar bifurcation diagrams:



Bifurcation diagrams for (a) $\beta = 1/2$ and (b) $\beta = 2$; $\nu = 0.1$, $\omega_1 = 1$, $\omega_2 = 1.7$; ---- stable, --- unstable

- Stability of steady-solutions

Variation of the amplitude equations:

$$\begin{pmatrix} \delta\dot{a}_1 \\ \delta\dot{a}_2 \end{pmatrix} = \mathbf{J}_s \begin{pmatrix} \delta a_1 \\ \delta a_2 \end{pmatrix}$$

where:

$$\mathbf{J}_s := \begin{pmatrix} \frac{\mu}{2} - \frac{9}{8} b\omega_1^2 (1+\beta) a_{1s}^2 - \frac{3}{4}\beta b \omega_2^2 a_{2s}^2 & -\frac{3}{2}\beta b \omega_2^2 a_{1s} a_{2s} \\ -\frac{3}{2}\beta b \omega_1^2 a_{1s} a_{2s} & \frac{\nu}{2} - \frac{9}{8} b\omega_2^2 (1+\beta) a_{2s}^2 - \frac{3}{4}\beta b \omega_1^2 a_{1s}^2 \end{pmatrix}$$

is the Jacobian evaluated at the steady-solution *s*.
In order that *s* is (asymptotically) stable, both the eigenvalues of $\mathbf{J}_s$ must have negative real part.

For each solution:

- Trivial solution (s=T): $\mathbf{J}_T = \mathrm{diag}[\mu/2, \nu/2]$, i.e. the trivial solution is stable in the third quadrant and unstable elsewhere;

- Periodic solutions (s=P_1,P_2):

$$\mathbf{J}_{P_1} = \mathrm{diag}[-\mu, \frac{\nu}{2} - \frac{\beta\mu}{1+\beta}], \quad \mathbf{J}_{P_2} = \mathrm{diag}[-\nu, \frac{\mu}{2} - \frac{\beta\nu}{1+\beta}]$$

An eigenvalue is always negative; the other vanishes at the straight lines r_2 and r_1. The P_1-solution is stable *below* r_2, and the P_2-solution is stable *above* r_1 .

(continue)

➢ Quasi-periodic solution (s=Q):

$$\mathrm{tr}[\mathbf{J}_Q] = -\frac{(1+\beta)(\mu+\nu)}{1+3\beta}, \quad \det[\mathbf{J}_Q] = \frac{2\beta\nu-(1+\beta)\mu}{\beta-1}\frac{2\beta\mu-(1+\beta)\nu}{1+3\beta}$$

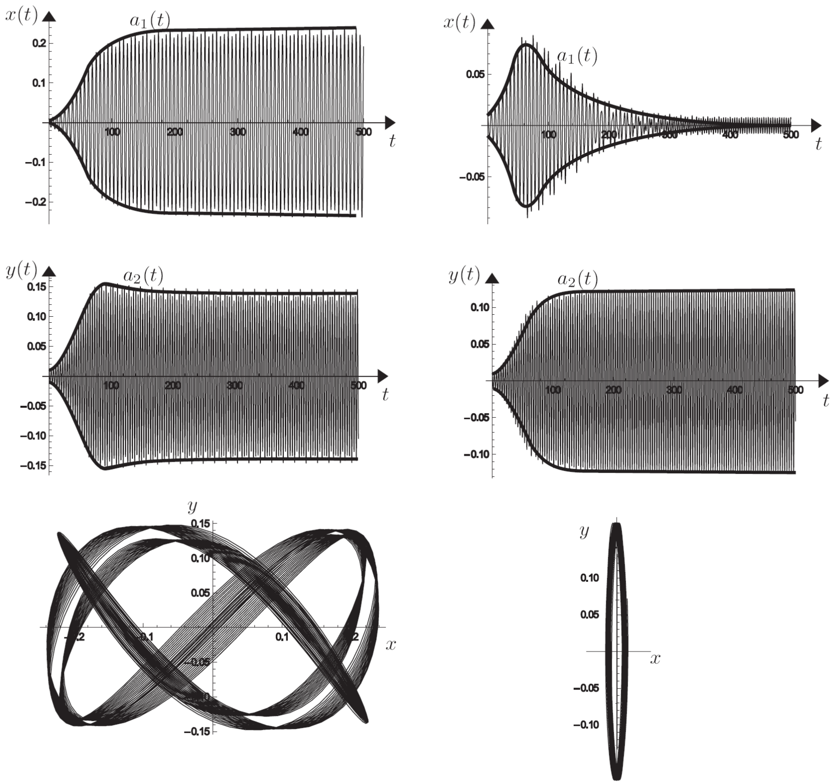
- ✓ For asymptotic stability $\mathrm{tr}[\mathbf{J}_Q]<0, \det[\mathbf{J}_Q]>0$ simultaneously.
- ✓ $\mathrm{tr}[\mathbf{J}_Q]<0$ in any points of the existence domain;
- ✓ $\det[\mathbf{J}_Q]=0$ at r_1 and r_2; inside the domain:

  $\det[\mathbf{J}_Q]>0$ when $\beta<1$ (Q-solution stable),

  $\det[\mathbf{J}_Q]<0$ when $\beta>1$ (Q-solution unstable).

■ Numerical integrations

Projection of the orbits onto the bi-dimensional (*x*, *y*) configuration-plane. Parameters are in the *Q*-region. Comparison between numerical solutions of the original equations and the RAME.



Numerical solutions for (*x*, *y*)-coordinates and (a_1, a_2)-amplitudes ; (a) $\beta = 1/2$, (b) $\beta = 2$; $\mu = 0.1, \nu = 0.1, \omega_1 = 1, \omega_2 = 1.7, c = 1$. Orbits refer to the steady-regime.

3. STATIC-DYNAMIC INTERACTION: THE DIVERGENCE-HOPF BIFURCATION

The MSM can be used also for multiple static-dynamic bifurcations. An example is given here.

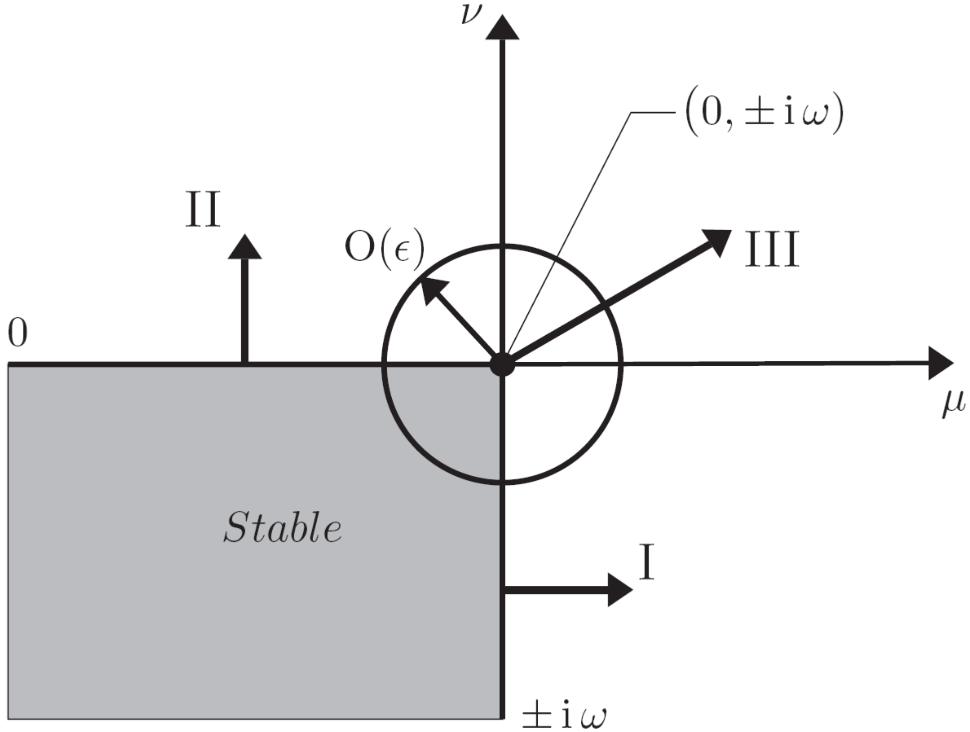
■ EXAMPLE: TWO COUPLED RAYLEIGH-DUFFING OSCILLATORS, BOTH UNSTABLE

The x-oscillator undergoes a *dynamic bifurcation*, governed by the parameter μ; the y-oscillator, suffers a *static bifurcation*, governed by the parameter ν:

$$\begin{cases} \ddot{x} - \mu \dot{x} + \omega^2 x + b_1 \dot{x}^3 + c x^3 - b_0 (y-x)^2 (\dot{y} - \dot{x}) - c_0 (y-x)^3 = 0 \\ \ddot{y} + \xi \dot{y} - \nu y + b_2 \dot{y}^3 + c y^3 + b_0 (y-x)^2 (\dot{y} - \dot{x}) + c_0 (y-x)^3 = 0 \end{cases}$$

where $\xi = \mathrm{O}(1) > 0$.

- Discussion on stability:



Linear stability diagram for the two Rayleigh-Duffing coupled oscillators undergoing divergence-Hopf bifurcation..

- Rescaling:

$$(\mu,\nu)\to(\varepsilon\mu,\varepsilon\nu)\,, \quad (x,y)\to(\varepsilon^{1/2}x,\varepsilon^{1/2}y)$$

- Series expansions:

$$\begin{pmatrix} x(t;\varepsilon) \\ y(t;\varepsilon) \end{pmatrix} = \begin{pmatrix} x_0(t_0,t_1,t_2,\cdots) \\ y_0(t_0,t_1,t_2,\cdots) \end{pmatrix} + \varepsilon \begin{pmatrix} x_1(t_0,t_1,t_2,\cdots) \\ y_1(t_0,t_1,t_2,\cdots) \end{pmatrix} + \varepsilon^2 \begin{pmatrix} x_2(t_0,t_1,t_2,\cdots) \\ y_2(t_0,t_1,t_2,\cdots) \end{pmatrix} + \cdots$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \mathrm{d}_0 + \varepsilon\,\mathrm{d}_1 + \varepsilon^2\,\mathrm{d}_2 + \cdots, \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2} = \mathrm{d}_0^2 + 2\varepsilon\,\mathrm{d}_0\,\mathrm{d}_1 + \varepsilon^2\left(\mathrm{d}_1^2 + 2\,\mathrm{d}_0\,\mathrm{d}_2\right) + \cdots$$

where $t_k := \varepsilon^k t_k$ and $\mathrm{d}_k := \partial/\partial t_k$.

- Perturbation equations:

$$\varepsilon^0 : \begin{cases} d_0^2\, x_0 + \omega^2 x_0 = 0 \\ d_0^2\, y_0 + \xi\, d_0 y_0 = 0 \end{cases}$$

$$\varepsilon^1 : \begin{cases} d_0^2\, x_1 + \omega^2 x_1 = -2\, d_0 d_1 x_0 + \mu\, d_0 x_0 - b_1 (d_0 x_0)^3 - c x_0^3 \\ \qquad + b_0 (y_0 - x_0)(d_0 y_0 - d_0 x_0)^2 + c_0 (y_0 - x_0)^3 \\ d_0^2\, y_1 + \xi\, d_0 y_1 = -2\, d_0 d_1 y_0 - \xi\, d_1 y_0 + \nu\, y_0 - b_2 (d_0 y_0)^3 - c y_0^3 \\ \qquad - b_0 (y_0 - x_0)(d_0 y_0 - d_0 x_0)^2 - c_0 (y_0 - x_0)^3 \end{cases}$$

- Generating solution:

$$\begin{cases} x_0 = A_1 \left(t_1, t_2, \ldots\right) e^{i\omega t_0} + c.c. \\ y_0 = a_2 \left(t_1, t_2, \ldots\right) \end{cases}$$

where $A_1 \in \mathbb{C}, a_2 \in \mathbb{R}$. The *x*-oscillator experiences a harmonic motion, slowly modulated; the *y*-oscillator rests in a non-trivial equilibrium position, also modulated.

- ε -order:
  - ➢ equations:

$$\begin{cases} \mathrm{d}_0^2\, x_1 + \omega^2 x_1 = f_{1,1}\, \mathrm{e}^{i\omega t_0} + NRT + c.c. \\ \mathrm{d}_0^2\, y_1 + \xi\, \mathrm{d}_0\, y_1 = f_{2,0} + (NRT + c.c.) \end{cases}$$

where the resonant excitations terms are:

$$f_{1,1} := -2i\omega\, \mathrm{d}_1 A_1 + i\omega\mu A_1$$
$$-[3(c+c_0) + ib_0\omega + 3ib_1\omega^3] A_1^2 \overline{A}_1 - (3c_0 + ib_0\omega) A_1 a_2^2$$
$$f_{2,0} := -\xi\, \mathrm{d}_1 a_2 + \nu A_2 - (c+c_0) a_2^3 - 6c_0 A_1 \overline{A}_1 a_2$$

- Removing secular terms requires: $f_{1,1}=0$ and $f_{2,0}=0$, from which:

$$\mathrm{d}_1 A_1 = \frac{1}{2}\mu A_1 + \frac{1}{2}[\frac{3}{\omega} i(c+c_0) - b_0 - 3b_1\omega^2] A_1^2 \overline{A}_1 - (3c_0 + ib_0\omega) A_1 a_2^2$$

$$\mathrm{d}_1 a_2 = \frac{1}{\xi}[\nu A_2 - (c+c_0) a_2^3 - 6c_0 A_1 \overline{A}_1 a_2]$$

- Parameter reabsorbing:

By multiplying the equations by $\varepsilon^{3/2}$ and using $\varepsilon^{1/2} A_1 \to A_1, \varepsilon^{1/2} a_2 \to a_2$, together with $\varepsilon(\mu,\nu) \to (\mu,\nu), \varepsilon\,\mathrm{d}_1 \to \mathrm{D}$, and by expressing A_1 in the polar form, three real bifurcation equations follows.

➢ Two amplitude-equations:

$$\begin{cases} \dot{a}_1 = \frac{1}{2}\mu a_1 - \frac{1}{8}(b_0 + 3b_1\omega_1^2)a_1^3 - \frac{1}{2}b_0 a_1 a_2^2 \\ \dot{a}_2 = \frac{1}{\xi}\nu a_2 - \frac{1}{\xi}(c + c_0)\omega_2^2 a_2^3 - \frac{3}{2\xi}c_0 a_1^2 a_2 \end{cases}$$

➢ One phase equation:

$$a_1\dot{\theta}_1 = \frac{3}{8}\frac{c + c_0}{\omega}a_1^3 + \frac{3}{2}\frac{c_0}{\omega}a_1 a_2^2$$

❑ **Note:** The amplitude equations governing the non-resonant double-Hopf bifurcation and the divergence-Hopf bifurcations have the same (normal) form.