

Equazioni lineari di Eulero

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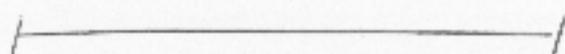
$$x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \dots + a_0 y = g(x)$$

$$a_0, \dots, a_{n-1} \in \mathbb{R}$$

negli intervalli $(0, +\infty)$ oppure $(-\infty, 0)$ si riconosce e un'eq. a coefficienti costanti sostituendo $x = e^t$ per $t \in (0, +\infty)$

$$x = -e^t \text{ per } t \in (-\infty, 0).$$

Oss per $x > 0$, integrali particolari del tipo $y = x^a$.



Esercizio (pagoni solso, p. 281 n. 13 (c))

Trovare l'integrale generale di

$$x^3 y''' + x y' - y = \frac{\ln x}{x} \quad x > 0$$

\Rightarrow uso la sostituzione $x = e^t$

con la sostituzione $x = e^t$, $t = \log x$

$$z(t) = y(e^t) \quad \text{avremo} \quad \boxed{y(x)} = z(\log x)$$

$$y(x) = z(\log x)$$

$$\frac{d}{dx} y(x) = z' \cdot \frac{1}{x} = \frac{d}{dt} z(\log x) \cdot \frac{1}{x}$$

$$\begin{aligned} \frac{d^2}{dx^2} y(x) &= z'' \cdot \frac{1}{x} \cdot \frac{1}{x} + z' \left(-\frac{1}{x^2}\right) \\ &= z'' \cdot \frac{1}{x^2} - z' \frac{1}{x^2} = (z'' - z') \frac{1}{x^2} \end{aligned}$$

$$\frac{d^3}{dx^3} y = \left(z''' \cdot \frac{1}{x} - z'' \cdot \frac{1}{x} \right) \frac{1}{x^2} +$$

$$2(z'' - z') \frac{1}{x^3} =$$

$$= (z''' - 3z'' + 2z') \frac{1}{x^3}$$

$$\Rightarrow x^3 y''' + x y' - y = \frac{\log x}{x} \text{ diventa}$$

$$x^3 (z''' - 3z'' + 2z') \frac{1}{x^3} + x \left(z' \cdot \frac{1}{x} \right) - z = te^{-t} \quad (*)$$

$$\bullet \quad z''' - 3z'' + 3z' - z = 0$$

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

$$(\lambda - 1)^3 = 0 \quad \Rightarrow \lambda = 1 \text{ mult. } 3$$

$$z_0(t) = c_1 e^t + c_2 t e^t + c_3 t^2 e^t$$

(3)

$$y_0(x) = z(\log x) =$$

$$= c_1 x + c_2 x \log x + c_3 x (\log x)^2$$

→ devo risolvere il problema non omogeneo
uso la variazione delle costanti

cerco la soluzione di (*) nella forma

$$z(t) = c_1(t) e^t + c_2(t) t e^t + c_3(t) t^2 e^t$$

devo trovare $c_1(t)$, $c_2(t)$, $c_3(t)$:

• calcolo

$$z'(t) = c_1' e^t + c_1 e^t + c_2' t e^t + c_2 (e^t + t e^t) + \\ + c_3' t^2 e^t + c_3 (2t e^t + t^2 e^t)$$

completamento

$$\boxed{c_1' e^t + c_2' t e^t + c_3' t^2 e^t = 0}$$

$$\Rightarrow z'(t) = c_1 e^t + c_2 (1+t) e^t + c_3 (2t+t^2) e^t$$

• ~~calcolo~~

$$z''(t) = c_1' e^t + c_1 e^t + c_2' (1+t) e^t + c_2 (e^t + (1+t) e^t) + \\ + c_3' (2t+t^2) e^t + c_3 ((2+2t) e^t + (2t+t^2) e^t)$$

completazione 2

(4)

$$\boxed{c_1' e^t + c_2' (1+t) e^t + c_3' (2t+t^2) e^t = 0}$$

$$\Rightarrow z''(t) = c_1 e^t + c_2 (2+t) e^t + c_3 (2+4t+t^2) e^t$$

• calcolo

$$z'''(t) = c_1' e^t + c_2 e^t + c_2' (2+t) e^t + c_2 (e^t + (2+t) e^t) + \\ + c_3' (2+4t+t^2) e^t + c_3 ((4+2t) e^t + (2+4t+t^2) e^t)$$

sostituisco z, z', z'', z''' in

$$z''' - 3z'' + 3z' - z = t \cdot e^t$$

$$\begin{aligned} & \left[c_1' e^t + c_2' (2+t) e^t + c_3' (2+4t+t^2) e^t + \right. \\ & \left. + c_1 e^t + c_2 (3+t) e^t + c_3 (6+6t+t^2) e^t \right] + \\ & + 3 \left[c_1 e^t + c_2 (1+t) e^t + c_3 (2t+t^2) e^t \right] + \\ & - \left[c_1 e^t + c_2 t e^t + c_3 t^2 e^t \right] \\ & - 3 \left[c_1 e^t + c_2 (2+t) e^t + c_3 (2+4t+t^2) e^t \right] = t e^{-t} \end{aligned}$$

→ i termini con c_1, c_2, c_3 si cancellano

rimane:

$$\boxed{c_1' e^t + c_2' (2+t) e^t + c_3' (2+4t+t^2) e^t = t e^{-t}}$$

completazione 3

Metto a sistema le 3 condizioni

$$\begin{cases} -c_1' e^t + c_2' t e^t + c_3' t^2 e^t = 0 \\ c_1' e^t + c_2' (1+t) e^t + c_3' (2t+t^2) e^t = 0 \\ c_1' e^t + c_2' (2+t) e^t + c_3' (2+4t+t^2) e^t = t e^{-t} \end{cases}$$

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$$\begin{cases} c_1' + c_2' t + c_3' t^2 = 0 \\ c_1' + c_2' (1+t) + c_3' (2t+t^2) = 0 \\ c_1' + c_2' (2+t) + c_3' (2+4t+t^2) = t e^{-2t} \end{cases}$$

$$(2)-(1) \Rightarrow \boxed{c_2'} + 2c_3' t = 0 \Rightarrow c_2' = -2c_3' t$$

$$(1) \Rightarrow \boxed{c_1'} = -c_2' t - c_3' t^2 = +2c_3' t^2 - c_3' t^2 = c_3' t^2$$

$$(3) \Rightarrow c_3' t^2 - 2t c_3' (2+t) + c_3' (2+4t+t^2) = t e^{-2t}$$

$$\Rightarrow c_3' t^2 - 4t c_3' - 2t^2 c_3' + 2c_3' + 4t c_3' + t^2 c_3' = t e^{-2t}$$

$$\Rightarrow 2c_3' = t e^{-2t}$$

Quindi

$$\begin{cases} c_1' = \frac{t^3}{2} e^{-2t} \\ c_2' = -t^2 e^{-2t} \\ c_3' = \frac{t}{2} e^{-2t} \end{cases}$$

Ora calcolo $c_1(t), c_2(t), c_3(t)$

$$c_3(t) = \int \frac{t}{2} e^{-2t} dt = \dots = -\frac{e^{-2t}}{4} \left(t + \frac{1}{2} \right) + k_3$$

$$c_2(t) = \int t^2 e^{-2t} = \dots = \frac{e^{-2t}}{2} \left(t^2 + t + \frac{1}{2} \right) + k_2$$

$$c_1(t) = \int \frac{t^3}{2} e^{-2t} dt = \dots = \frac{e^{-2t}}{4} \left(-t^3 - \frac{3}{2} t^2 - \frac{3}{2} t - \frac{3}{4} \right) + k_1$$

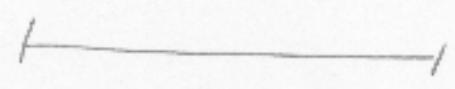
Una volta $z(t) + c_1(t) e^t + c_2(t) t e^t + c_3(t) t^2 e^t$ e

trovo le soluzioni di *

~~$$z(t) = \frac{e^{-2t}}{4} \left(-t^3 - \frac{3}{2} t^2 - \frac{3}{2} t - \frac{3}{4} \right) + k_1 e^t +$$~~

$$+ \frac{e^{-t}}{2} t \left(t^2 + t + \frac{1}{2} \right) + k_2 t e^t +$$

$$- \frac{e^{-t}}{4} t^2 \left(t + \frac{1}{2} \right) + k_3 t^2 e^t$$



Infine trovo la soluzione $y(x)$ del problema iniziale si accorgo che $t = \log x$:

$$y(x) = -\frac{x}{4} \left(-(\log x)^3 - \frac{3}{2} (\log x)^2 - \frac{3}{2} (\log x) - \frac{3}{4} \right) +$$

$$+ k_1 x - \frac{x}{2} \log x \left((\log x)^2 + \log x + \frac{1}{2} \right) + k_2 x \log x +$$

$$+ \frac{x (\log x)^2}{4} \left(\log x + \frac{1}{2} \right) + k_3 x (\log x)^2.$$

(Potrai scrivere raccogliendo!)