Applied Partial Differential Equations (MathMods)
Exercise sheet 2

Fully nonlinear first order equations:

- Do the following exercises from Salsa's book [1]: **4.18, 4.19, 4.20.**

In addition, do the following exercises:

1. Solve the equation
$$xu_x + yu_y + u_x u_y - u = 0,$$
with initial condition $u(x, -x) = 1$.
2. For the equation
$$u_y = u_x^3,$$
find the solution for $u(x, 0) = 2x^{3/2}$ (*Answer:* $2x^{3/2}(1 - 27y)^{-1/2}$.)
3. Find a solution to the equation
$$u = xu_x + yu_y + \tfrac{1}{2}(u_x^2 + u_y^2),$$
with initial condition $u(x, 0) = \frac{1}{2}(1 - x^2)$.
4. Find the solution to the following initial-value problem
$$u_t + uu_x = 1,$$
$$u(x, 0) = -\tfrac{1}{2}x.$$
Find the characteristic curves (draw a picture) and give an explicit formula for the solution. Does your solution exist for all $(x, t) \in \mathbb{R} \times [0, +\infty)$? Why or why not?

Conservation laws:

- Do the following exercises from Salsa's book [1]: **4.5, 4.6, 4.9.**

In addition, do the following exercises:

5. Consider Burgers' equation
$$u_t + \left(\tfrac{1}{2}u^2\right)_x = 0, \tag{1}$$
with initial condition
$$u_0(x) = \begin{cases} u_R, & x > 0, \\ u_L, & x < 0, \end{cases} \tag{2}$$
where $u_L < u_R$. Show that
$$u(x,t) = \begin{cases} u_L, & x < s_m t, \\ u_m, & s_m t \le x \le u_m t, \\ x/t, & u_m t \le x \le u_R t, \\ u_R, & x > u_R t, \end{cases}$$

is a weak solution to the problem for any value u_m satisfying $u_L \leq u_m \leq u_R$ and where $s_m := \frac{1}{2}(u_L + u_m)$. Draw the characteristics and draw a scheme of the solution. Find a class of solutions with three discontinuities.

6. For the same Burgers' equation (1), solve the initial value problem with initial condition $u(x, 0) = x$. Does the solution exist for all times? Why or why not?

References

[1] S. Salsa, *Partial differential equations in action. From modelling to theory*, Universitext, Springer-Verlag Italia, Milan, 2008.