**Applied Partial Differential Equations (MathMods)**
**Second Midterm Exam: Solution**

**1:** Let $x \in \mathbb{R}^3$ and $R > 0$ be arbitrary. Since $u$ is harmonic in the whole space $\mathbb{R}^3$ it satisfies the mean value property in the ball $B_R(x)$. Therefore,

$$\begin{aligned} 0 \le |u(x)| &= \left| \frac{3}{4\pi R^3} \int_{B_R(x)} u(y)\,dy \right| \\ &\le \frac{3}{4\pi R^3} \int_{B_R(x)} |u(y)|\,dy \\ &\le \frac{3}{4\pi R^3} \left( \int_{B_R(x)} dy \right)^{1/2} \left( \int_{B_R(x)} |u(y)|^2\,dy \right)^{1/2} \\ &\le \frac{3}{4\pi R^3} \left( \frac{4\pi R^3}{3} \right)^{1/2} \left( \int_{\mathbb{R}^3} |u(y)|^2\,dy \right)^{1/2} \\ &= \frac{\sqrt{3M}}{2\sqrt{\pi}} \frac{1}{R^{3/2}} \longrightarrow 0, \end{aligned}$$

as $R \to +\infty$. Therefore $u(x) = 0$ for all $x \in \mathbb{R}^3$.

**2(a):** By Green's formula: $\int_D \Delta u\,dxdy = \int_{\partial D} \partial u/\partial n\,dS$. Therefore.

$$\int_D k\,dxdy = k\pi a^2 = \int_0^{2\pi} \cos^2\theta\,a d\theta = \frac{a}{2}\int_0^{2\pi} (1+\cos 2\theta)\,d\theta = \pi a.$$

Hence, $k = 1/a$.

**2(b):** Using the *ansatz*:

$$u = A_0(r) + \sum_{n=1}^{\infty} A_n(r)\cos(n\theta), \quad u_r = A_0'(r) + \sum_{n=1}^{\infty} A_n'(r)\cos(n\theta), \quad u_{rr} = A_0''(r) + \sum_{n=1}^{\infty} A_n''(r)\cos(n\theta),$$

$$u_\theta = -\sum_{n=1}^{\infty} nA_n(r)\sin(n\theta), \qquad u_{\theta\theta} = -\sum_{n=1}^{\infty} n^2 A_n(r)\cos(n\theta),$$

Upon substitution,

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = A_0''(r) + \frac{1}{r}A_0'(r) + \sum_{n=1}^{\infty}\left( A_n''(r) + \frac{1}{r}A_n'(r) - \frac{n^2}{r^2}A_n(r) \right)\cos(n\theta) = \frac{1}{a},$$

and we obtain a hierarchy of equations:

$$A_0''(r) + \frac{1}{r}A_0'(r) = \frac{1}{a},$$

$$A_n''(r) + \frac{1}{r}A_n'(r) - \frac{n^2}{r^2}A_n(r) = 0, \quad n = 1, 2, \ldots$$

First we solve the equation for $A_0$. Multiply by $r$ so that $rA_0'' + A_0' = \frac{d}{dr}(rA_0') = \frac{r}{a}$. Thus,

$$A_0'(r) = \frac{r}{2a} + \frac{C_1}{r}.$$

Integrating $A_0(r) = \frac{r^2}{4a} + C_1 \log r + C_2$, for some constants $C_1, C_2$. Since the solution is bounded at $r = 0$ we have $C_1 = 0$. Thus:

$$A_0(r) = \frac{r^2}{4a} + C_2.$$

Next we solve the equation for $A_n$: $A_n'' + \frac{1}{r}A_n' - \frac{n^2}{r^2}A_n = 0$. Proposing a solution of the form $A_n = r^\alpha$ we arrive at $r^{\alpha-2}(\alpha^2 - n^2) = 0$, yielding $\alpha = \pm n$. Therefore the solutions have the form

$$A_n(r) = K_n r^n + \frac{C_n}{r^n},$$

with $C_n, K_n$ constants. Since the solution is bounded at $r = 0$ we get $C_n = 0$. Therefore the solution has the general form:

$$u(r,\theta) = \frac{r^2}{4a} + C_2 + \sum_{n=1}^{\infty} K_n r^n \cos(n\theta).$$

From the boundary condition we obtain

$$\frac{\partial u}{\partial n}_{|r=a} = u_r(a,\theta) = \frac{1}{2} + \sum_{n=1}^{\infty} K_n n a^{n-1}\cos(n\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta).$$

This implies that $K_1 = 0$, $2K_2 a = 1/2$ and $K_n = 0$ for all $n \geq 3$. Therefore the solution is given by

$$u(r,\theta) = \frac{r^2}{4a}\big(1 + \cos 2\theta\big) + C = \frac{r^2}{2a}\cos^2\theta + C, \tag{1}$$

where $C$ is an arbitrary constant. We verify that this is a solution by computing:

$$\Delta u = \frac{2}{a}\cos^2\theta + \frac{1}{a}(1 - 2\cos^2\theta) = \frac{1}{a},$$

and $(\partial u/\partial n)_{|r=a} = u_r(a,\theta) = \cos^2\theta$.

**2(c):** Suppose that $v$ is another solution to the same problem. Then $w = u - v$ is a solution to $\Delta w = 0$ in $D$, and $\partial w/\partial n = 0$ at $\partial D$. By Green's formula we have

$$0 = \int_D w\Delta w\, dxdy = \int_{\partial D} w\frac{\partial w}{\partial n}\, ds - \int_D |\nabla w|^2\, dxdy = -\int_D |\nabla w|^2\, dxdy.$$

Therefore $|\nabla w| = 0$ in $D$, which means that $w$ is constant in $D$. Therefore the solution is unique up to a constant. In view of the form of $u$ (equation (1)) with $C$ arbitrary we conclude that all solutions have the form (1).

**3(a):** We have two cases: either $x \geq ct > 0$ (region I), or $0 < x < ct$ (region II). In region I, since $x + ct > x - ct \geq 0$, the solution is given by D'Alembert's formula:

$$u(x,t) = \frac{1}{2}(f(x+ct) + f(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\, dy.$$

In the region II, consider a point $P = (x,t)$ such that $0 < x < ct$, and draw the characteristic rhomboid $PQRS$ like in Figure 1. Since $PQ$ is characteristic with $x_Q = 0$ we have $x = c(t - t_Q)$. Therefore $Q = (0, t - x/c)$, with $t_Q = t - x/c > 0$. By the boundary condition: $u(Q) = h(t - x/c)$. Now, take $QR$ characteristic. Thus, since $t_R = 0$ we have $x_R = ct - x > 0$, and $R = (ct - x, 0)$. By the initial condition

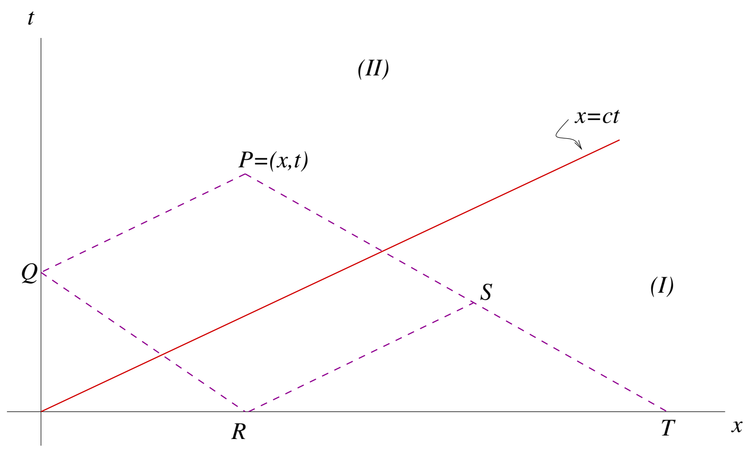


FIGURE 1. Characteristic rhomboid $PQRS$ for $P$ in region II.

$u(R) = f(ct-x)$. In the same fashion, $S = (ct, x/c) = (x_S, t_S)$ and since $S$ belongs to region I we have, by D'Alembert's formula:

$$\begin{aligned} u(S) &= \frac{1}{2}(f(x_S + ct_S) + f(x_S - ct_S)) + \frac{1}{2c}\int_{x_S - ct_S}^{x_S + ct_S} g(y)\,dy \\ &= \frac{1}{2}(f(x+ct) + f(ct-x)) + \frac{1}{2c}\int_{ct-x}^{ct+x} g(y)\,dy. \end{aligned}$$

By the parallelogram theorem we have: $u(P) = u(Q) + u(S) - u(R)$. Hence,

$$u(P) = u(x,t) = h(t - x/c) + \frac{1}{2}(f(x+ct) - f(ct-x)) + \frac{1}{2c}\int_{ct-x}^{ct+x} g(y)\,dy.$$

The full solution is given by:

$$u(x,t) = \begin{cases} \frac{1}{2}(f(x+ct) + f(x-ct)) + \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} g(y)\,dy, & 0 < ct \le x, \\ h(t-x/c) + \frac{1}{2}(f(x+ct) - f(ct-x)) + \dfrac{1}{2c}\displaystyle\int_{ct-x}^{ct+x} g(y)\,dy, & 0 < x < ct. \end{cases} \tag{2}$$

**3(b):** The solution (2) is clearly of class $C^1$ in the interior of the regions I and II because $f$, $g$ and $h$ are continuously differentiable functions. To see what happens in the line $x = ct$ we compute:

$$u_x = \begin{cases} \frac{1}{2}(f'(x+ct) + f'(x-ct)) + \frac{1}{2c}(g(x+ct) - g(x-ct)), & 0 < ct < x, \\ -\frac{1}{c}h'(t-x/c) + \frac{1}{2}(f'(x+ct) + f'(ct-x)) + \frac{1}{2c}(g(ct+x) + g(ct-x)), & 0 < x < ct. \end{cases}$$

$$u_t = \begin{cases} \frac{c}{2}(f'(x+ct) - f'(x-ct)) + \frac{1}{2}(g(x+ct) + g(x-ct)), & 0 < ct < x, \\ h'(t-x/c) + \frac{c}{2}(f'(x+ct) - f'(ct-x)) + \frac{1}{2}(g(ct+x) - g(ct-x)), & 0 < x < ct. \end{cases}$$

We observe that

$$\tfrac{1}{2}(f(2ct) + f(0)) + \frac{1}{2c}\int_0^{2ct} g(y)\,dy = h(0) + \tfrac{1}{2}(f(2ct) - f(0)) + \frac{1}{2c}\int_0^{2ct} g(y)\,dy,$$

because $h(0) = f(0)$. Therefore $u$ is continuous at $x = ct$. Moreover, note that

$$\tfrac{1}{2}(f'(2ct)+f'(0))+\tfrac{1}{2c}(g(2ct)-g(0)) = -\tfrac{1}{c}h'(0)+\tfrac{1}{2}(f'(2ct)+f'(0))+\tfrac{1}{2c}(g(2ct)+g(0))$$

inasmuch as $h'(0) = g(0)$. Therefore $u_x$ is continuous at $x = ct$. Finally, notice that

$$\tfrac{c}{2}(f'(2ct)-f'(0))+\tfrac{1}{2}(g(2ct)+g(0)) = h'(0)+\tfrac{c}{2}(f'(2ct)-f'(0))+\tfrac{1}{2}(g(2ct)-g(0)),$$

again because $h'(0) = g(0)$. Therefore $u_t$ is continuous at $x = ct$. We conclude that the solution (2) is of class $C^1$ in the domain $x > 0$ and $t > 0$.

**3(c):** If $f(x) = x$, $g(x) = 1$ and $h(t) = t$ then $\frac{1}{2}(f(x+ct)+f(x-ct)) = x$, $\frac{1}{2}(f(ct+x)-f(ct-x)) = x$, $h(t-x/c) = t-x/c$, and

$$\frac{1}{2c}\int_{x-ct}^{x+ct} dy = t, \qquad \frac{1}{2c}\int_{ct-x}^{ct+x} dy = \frac{x}{c}.$$

Therefore the solution in region I is $u(x,t) = x+t$; in region II it is also $u(x,t) = x+t$, verifying the conclusion in (b).

**4:** By Kirchhoff's formula and Duhamel principle the solution is given by (in view of $f = 0$):

$$u(x,t) = \frac{1}{4\pi c^2 t}\int_{|x-y|=ct} g(y)\,dS_y + \frac{1}{4\pi}\int_0^t \frac{1}{c^2(t-s)}\int_{|x-y|=c(t-s)} h(y,s)\,dS_y\,ds$$

Substituting $g(y) = y_2$ and $h(y,s) = s$ we obtain:

$$\begin{aligned}\frac{1}{4\pi c^2 t}\int_{|x-y|=ct} y_2\,dS_y &= \frac{t}{4\pi}\int_{|\eta|=1} x_2 + ct\eta_2\,dS_\eta \\ &= \frac{tx_2}{4\pi}\int_{|\eta|=1} dS_\eta + \frac{ct^2}{4\pi}\int_{|\eta|=1}\eta_2\,dS_\eta = tx_2,\end{aligned}$$

because $\int_{|\eta|=1}\eta_2 dS_\eta = 0$ (odd function integrated in a symmetric domain; you can verify this in cartesian coordinates or in spherical coordinates[1]). Also,

$$\frac{1}{4\pi}\int_0^t \frac{1}{c^2(t-s)}\int_{|x-y|=c(t-s)} s\,dS_y\,ds = \frac{1}{4\pi}\int_0^t s(t-s)\int_{|\eta|=1} dS_\eta\,ds = \int_0^t s(t-s)\,ds = \frac{t^3}{6}.$$

The solution is:

$$u(x,t) = tx_2 + \frac{t^3}{6}. \tag{3}$$

It clearly satisfies the initial conditions: $u(x,0) = 0$ and $u_t(x,0) = x_2$. It solves the equation because $\Delta u = 0$ and $u_{tt} = t$.

**5:** The solution is given by D'Alembert's formula:

$$u(x,t) = \tfrac{1}{2}(f(x+ct)+f(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy.$$

Since $f$ and $g$ have compact support, they (and all their derivatives) are zero outside an interval $|x| \le R$. Therefore, for any fixed $t > 0$ the solution also has

[1]For example, in spherical coordinates $\int_{|\eta|=1}\eta_2 dS_\eta = \int_0^\pi\int_0^{2\pi}\sin^2\phi\sin\theta\,d\theta\,d\phi = 0$.

compact support in the $x$ variable. Actually, if $|x| > R + ct$ for fixed $t > 0$, then $u$, $u_x$ and $u_t$ are zero. Then we compute the derivative of the total energy:

$$\begin{aligned}\frac{dE(t)}{dt} &= \frac{1}{2}\frac{d}{dt}\int_{-\infty}^{+\infty} u_t^2 + c^2 u_x^2\,dx = \int_{-\infty}^{+\infty} u_t u_{tt} + c^2 u_x u_{xt}\,dx \\ &= c^2\int_{-\infty}^{+\infty} u_t u_{xx}\,dx + c^2 (u_x u_t)\big|_{x=-\infty}^{x=+\infty} - c^2\int_{-\infty}^{+\infty} u_t u_{xx}\,dx \\ &= c^2 (u_x u_t)\big|_{x=-\infty}^{x=+\infty} = 0,\end{aligned}$$

after integrating by parts and in view that $u_t$ and $u_x$ have compact support for each $t > 0$ fixed. This shows that the energy is conserved:

$$E(t) = E(0),$$

for all $t \geq 0$. To show the equipartition of energy we have by D'Alembert's formula that

$$\begin{aligned}u_x &= \tfrac{1}{2}(f'(x+ct) + f'(x-ct)) + \frac{1}{2c}(g(x+ct) - g(x-ct)), \\ u_t &= \tfrac{c}{2}(f'(x+ct) - f'(x-ct)) + \frac{1}{2}(g(x+ct) + g(x-ct)),\end{aligned}$$

Therefore,

$$\begin{aligned}u_t^2 - c^2 u_x^2 &= (u_t + cu_x)(u_t - cu_x) \\ &= (cf'(x+ct) + g(x+ct))\,(-cf'(x-ct) + g(x-ct)).\end{aligned}$$

Now, since $f'$ and $g$ have support in $|x| \leq R$, then taking either $|x - ct| > R$ or $|x + ct| > R$ we have $f' = 0$ and $g = 0$. Then, taking $t$ sufficiently large, more precisely $t > T := R/c$, we have that for each fixed $x \in \mathbb{R}$, either $|x - ct| > R$ or $|x + ct| > R$. Indeed: if $x \geq 0$ then $t > T = R/c \geq (R - x)/c$ and thus $|x + ct| \geq x + ct > R$. On the other hand, if $x < 0$ then $t > T = R/c \geq (R + x)/c$, yielding $|x - ct| \geq x - ct > R$. Thus, for all times greater than $T = R/c$ we have $u_t^2 = c^2 u_x^2$. This implies the equipartition of energy:

$$E_{\mathrm{cin}}(t) = \tfrac{1}{2}\int_{-\infty}^{+\infty} u_t^2\,dx = E_{\mathrm{pot}}(t) = \tfrac{1}{2}\int_{-\infty}^{+\infty} c^2 u_x^2\,dx,$$

for all times $t > T$.