SOLVING THE SYSTEM OF LINEAR ELASTICITY BY A SCHWARZ METHOD

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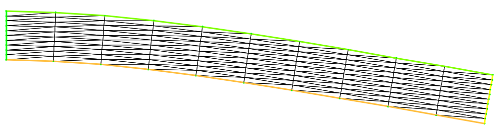
1 System of linear elasticity

Linear elasticity is a simplification of the more general nonlinear theory of elasticity and is a branch of continuum mechanics. The fundamental "linearizing" assumptions of linear elasticity are: infinitesimal strains or "small" deformations (or strains) and linear relationships between the components of stress and strain. In addition linear elasticity is only valid for stress states that do not produce yielding. These assumptions are reasonable for many engineering materials and engineering design scenarios. Linear elasticity is therefore used extensively in structural analysis and engineering design, often through the aid of finite element analysis.

In the following we recall some results concerning the system of linear elasticity, we will apply the variational approach to the resolution of system of equation of linear elasticity.

We start with the description of the physical model
Figure : Beam fixed on one side
Displacement under the gravity force



Let Ω be an open set of R^N, $f(x)$ a force which is a function from Ω to R^N and the unknown u displacement which is a function of Ω in R^N the mecanic modelisation involves the tensor of deformation $e(u)$ defined by

$e(u) = \frac{1}{2}(\nabla u + (\nabla u)^t) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})\ i = j = 1...N.$
This modelisation involves also the tensor of constraint σ

$$\sigma = 2\mu e(u) + \lambda tr(e(u))Id$$

where λ, μ are Lame coefficients of the material. For some thermodynamic reason λ, μ satisfy
$\mu > 0$ and $2\mu + N\lambda > 0$
Using the sum of all the forces in the solid we obtain :

$$-div(\sigma) = f \quad \text{in } \Omega$$

Using the fact that $tr(u) = divu$, we can deduce the following equation :

$$-\sum_{j=1}^{N} \frac{\partial}{\partial x_j}(\mu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) + \lambda(divu)\delta_{ij}) = f_i \text{ in } \Omega$$

With u_i, f_i the components of f and u in the canonical basis of R^N. Adding Dirichlet boundary condition we obtain the following boundary value problem :

$$\left\{ \begin{array}{c} -div(2\mu e(u) + \lambda tr(e(u))Id) = f \,\text{in}\, \Omega \\ u = 0 \,on\, \partial\Omega \end{array} \right. \tag{1}$$

The result below shows that the boundary value problem1 is well posed by using variational approach.

Theorem

There exists a unique weak solution u in $H_0^1(\Omega)^N$ of equations (1). We will present the main ingredients of the proof.

To find the variational formulation we multiply the equation by a smooth test function which vanish on the border of Ω, and we integrate by parts to obtain :

$$\int_\Omega \mu \sum_{j=1}^{N} (\frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j}) \frac{\partial v_i}{\partial x_j} dx + \int_\Omega \lambda divu \frac{\partial v_i}{\partial x_i} dx = \int_\Omega f_i v_i dx$$

Using some summation properties we obtain the variational formulation : find u in $H_0^1(\Omega)^N$ such that

$$\int_\Omega 2\mu e(u).e(v)dx + \int_\Omega \lambda divudivvdx = \int_\Omega fvdx$$

for all v in $H_0^1(\Omega)^N$

In order to apply the Lax-Milgram theorem we need to verify the coercivity of the bilinear functional $a(u,v) = \int_\Omega 2\mu e(u).e(v)dx + \int_\Omega \lambda divudivvdx$

We can show that

$$\int_\Omega 2\mu |e(u)|^2 dx + \int_\Omega \lambda |divu|^2 dx \geq \alpha \int_\Omega |e(v)|^2 dx$$

with $\alpha = \min(2\mu, (2\mu + N\lambda)) \succ 0$. Next we use Korn inequality which gives a constant $C \succ 0$ such that :

$$\int_\Omega |e(v)|^2 dx \geq C \int_\Omega |\nabla v|^2 dx$$

for all v in $H_0^1(\Omega)^N$.

Thirdly we use Poincare inequality which gives C $\succ$ 0 such that for all v in $H_0^1(\Omega)^N$ we have,

$$\int_\Omega |v|^2 dx \leq C \int_\Omega |\nabla v|^2 dx$$

Combining these inequalities we obtain the coercivity of a mean

$$\int_\Omega 2\mu |e(u)|^2 dx + \int_\Omega \lambda |divu|^2 dx \geq C ||v||^2_{H^1(\Omega)}$$

Applying Lax-Milgram theorem we obtain the existence and uniqueness of the solution to the variational formulation

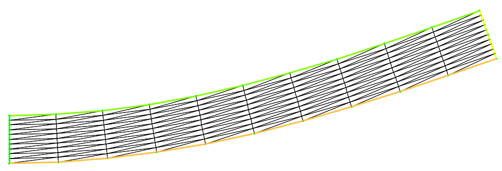
Lemma

LetΩ an open set of R^N for all v in $H_0^1(\Omega)^N$ we have :
$||\nabla v||_{L^2(\Omega)} \leq \sqrt{2} ||e(v)||_{L^2(\Omega)}$.

Since we have used Dirichlet boundary condition in pratice a part of the border can be free to move or some surface forces can be applied on the border these two cases are modelised by Neumann boundary condition,$\sigma n = g$ on$\partial\Omega$

Figure :Beam fixed on one side

Neumann condition on another



where g is a function in $L^2(\Omega)^N$ (the force applied on the border).

Now we consider a system of linear elasticity with mixed boundary conditions, Dirichlet and Neumann i.e

$$\begin{cases} -div(2\mu e(u) + \lambda tr(e(u))Id) = f \quad \text{in}\,\Omega \\ u = 0\,\text{on}\,\partial\Omega_D \\ \sigma n = g\,on\,\partial\Omega_N \end{cases} \tag{2}$$

where $(\partial\Omega_N, \partial\Omega_D)$ is a partition of $\partial\Omega$ of non zero measure. Existence and uniqueness can be proved using Korn Inequality.

Lemma (Korn Inequality)

Let Ω be open bounded and regular set of class C^1 of R^N. There exists a constant C$\succ 0$ such that for all function $v \in H^1(\Omega)^N$ we have

$||v||_{H^1(\Omega)} \leq$C $(||v||^2_{L^2(\Omega)} + ||e(v)||^2_{L^2(\Omega)})^{1/2}$.

The mecanic interpretation of Korn inequality is the following : the elastic energy proportional to the norm of the tensor of deformation $e(u)$ in $L^2(\Omega)$ controls the norm of the displacement u in $H^1(\Omega)^N$ up to the addition of the norm of u in $L^2(\Omega)$.

Theorem

Let Ω an open bounded connected regular set of class C^1 of R^N. Let $f \in L^2(\Omega)$ $g \in L^2(\partial\Omega_N)^N$ we define the space

V $= \{v \in H^1(\Omega)^N$ such that $v = 0$ on $\partial\Omega_D\}$

There exists a unique weak solution $u \in$ V of (2) which depends linearly on f and g

The solution of the variational problem can be interpreted as the minimization of an energy

Proposition

Let $j(v)$ the energy defined for all $v \in$ V by :

$$j(v) = \frac{1}{2}\int_\Omega (2\mu|e(v)|^2 + \lambda|divv|^2)dx - \int_\Omega f.vdx - \int_{\partial\Omega_N} g.vds$$

Let u be the unique solution of the variational formulation of (2), then$u$ is the unique minimum point of the above energy in V. Reciprocally if $u \in$ V is the minimum point of the energy $j(v)$ then u is the unique solution of the variational formulation.

2 Smith factorization applied to the linear elasticity

Smith factorization is an algebraic tool which allows to treat matrices with polynomial entries

We consider a matrix with polynomial entries in one variable

$$A(\lambda)=\begin{pmatrix} a_{11}(\lambda) & ... & a_{1n}(\lambda) \\ : & ... & : \\ a_{m1}(\lambda) & ... & a_{mn}(\lambda) \end{pmatrix}.$$

We recall the Smith factorization of a matrix with polynomial entries

Theorem Let n be an positive integer and A an invertible $n \times n$ matrix with polynomial entries with respect to the variable λ: $A = (a_{ij}(\lambda))_{1\leq i,j\leq n}$. Then, there exist matrices E, D and F with polynomial entries satisfying the following properties:

- $det(E)$ and $det(F)$ are constants,
- D is a diagonal matrix uniquely determined up to a multiplicative constant,
- $A = EDF$.

Here E and F are matrices, which operate on the rows resp. columns. The entries of the diagonal matrix $D = (d_{ij}(\lambda))$ are given by $d_{ii} = \phi_i/\phi_{i-1}$, where ϕ_i is the greatest common divisor of the determinants of all $i \times i$ sub matrices of A and $\phi_0 = 1$.

The Smith factorization is a classical tool in computer algebra and in control of ordinary differential equations. Since its use in scientific computing is rather new, we give here a few comments:

- Smith was an English mathematician of the end of the 19th century. He worked in number theory and considered the problem of factorizing matrices with integer entries. We gave here the polynomial version of his theorem in the special case where the matrix A is square and invertible but the result is more general and applies as well when the matrix$A$ is rectangular.
- One of the interest of the theorem is the following. By Cramer's formula, the inverse of A is in general a matrix with rational entries. By the Smith factorization, we have $A^{-1} = F^{-1}D^{-1}E^{-1}$. Since $det(E)$ and $det(F)$ are constants, the inverse of E and F are still matrices with polynomial entries in λ. The rational part of the inverse of$A$ is thus in D^{-1} which is an intrinsic diagonal matrix.
- The proof of the theorem is constructive and gives an algorithm for computing matrices E, D and F. As stated in the theorem, matrix D is intrinsic but matrices E and F are not unique.
- In the sequel, we write the system of linear elasticity as a matrix with partial differential operators entries applied to the unknown displacement. The direction normal to the interface of the subdomains is particularized and denoted by∂_x. Each partial differential operator is then considered as a polynomial in the "variable ∂_x" (e.g. Λ is related to ∂_x and Λ^2 to ∂_{xx}). It is then possible to apply the Smith factorization, see below.

We consider two elementary operations on the matrix : 1) permutation of rows (and columns), 2)multiply a row (or column) by a scalar polynomial and add it to another row (or column).

These transformations keep the matrix with polynomial entries and preserve (up to a sign) the determinant of the matrix.

For a matrix whose first entry is a non zero polynomial of minimal degree, we consider three possibilities :

1) there exists at least one entry in the first line or the first column that $a_{11}(\lambda)$ does not divide.

2) The first entry $a_{11}(\lambda)$ divides all the entries of the first line and of the first column and in addition to the first entry $a_{11}(\lambda)$, one of the entries of the first line and of the first column is not zero.

3) Except for the first entry $a_{11}(\lambda)$, all the entries of the first line and of the first column are zero and there exist at least one entry in the matrix that $a_{11}(\lambda)$ does not divide

4) Except for the firt entry $a_{11}(\lambda)$, all the entries of the first line and of the first column are zero and $a_{11}(\lambda)$ divides all the entries of the matrix.

The above cases are exclusive and cover all possible situations. We shall show that by elementary operations :

In case 1, it is possible to decrease the minimal degree of the matrix.

In case 3, it is possible to go to case 1.

In case 2, it is possible to go to case 3 or 4.

The generic situation is case 1. We propose that it is possible to decrease the minimal degree of the matrix. Suppose $a_{11}(\lambda)$ does not divide an entry of the first line (the argument would be similar for an entry of the first column), say a_{1j}. Then, perform the Euclidean division of a_{1j} by $a_{11}(\lambda)$:

$a_{1j}(\lambda)= b_j(\lambda)a_{11}(\lambda) + r_j(\lambda)$ where the degree of r_j is less than that of $a_{11}(\lambda)$. Then, multiply the first column by $-b_j$ and add the result to the jth column of A, so that the $j-th$ element of the first line is r_j. If r_j is not zero, permute the first and the $j-th$ columns so that r_j is the first coefficient of the matrix. Note that the minimal degree of A has decreased. Since the minimal degree of A is not negative, after a finite number of applications of this procedure, we are sure to leave case 1.

Suppose we are in case 3. Let $a_{ij}(\lambda)$ be a polynomial that $a_{11}(\lambda)$ does not divide. By adding the $i-th$ row to the first row, we go to case 1.

Suppose we are in case 2. For each $1\prec j\leq n$, we multiply the first column by a scalar polynomial $-a_{ij}(\lambda)/a_{11}(\lambda)$ and add it to the $j-th$ column. Then, all the coefficients of the first line are zero except for the first one. The coefficients of the first column are left unchanged by this operation.

Thus, for each $2\prec j\leq m$ multiplying the first line (which has only one non zero entry) by $-a_{j1}(\lambda)/a_{11}(\lambda)$ and adding it to the j-th line we cancel all the coefficients of the first column except for the first one. We are now thus either in case 3 or in case 4.

It is thus possible after a finite number of steps to go to case 4 and then apply the same procedure to submatrix $A(2:n,2:n)$.

The first equation of the system (1) in two dimension is given by :

$$S_2\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} (2\mu+\lambda)\partial_{xx}+\mu\partial_{yy} & \lambda\partial_{xy}+\mu\partial_{yx} \\ \mu\partial_{xy}+\lambda\partial_{yx} & \mu\partial_{xx}+(2\mu+\lambda)\partial_{yy} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

We transform this equations as follows :

we perform Fourier transform in the y-direction with the dual variable k,

we perform Laplace transform in the x-direction with dual variable Λ,

we obtain the following equation :

$$\begin{pmatrix} (2\mu+\lambda)\Lambda^2-k^2\mu & (\lambda+\mu)ik\Lambda \\ (\mu+\lambda)ik\Lambda & \mu\lambda^2-(2\mu+\lambda)k^2 \end{pmatrix}\begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = \hat{f}$$

Let A be the following matrix

$$A=\begin{pmatrix} (2\mu+\lambda)\Lambda^2-k^2\mu & (\lambda+\mu)ik\Lambda \\ (\mu+\lambda)ik\Lambda & \mu\Lambda^2-(2\mu+\lambda)k^2 \end{pmatrix}$$

The smith factorization of A is build as follows by :

Initially we are in case 1 we permutte the columns and reduce the degree of the first entry

$$\mathbf{A}_1 = \mathbf{A}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} i(\mu+\lambda)k\Lambda & (2\mu+\lambda)\Lambda^2-k^2\mu \\ \mu\Lambda^2-(2\mu+\lambda)k^2 & (\mu+\lambda)ik\Lambda \end{pmatrix}$$

$$\mathbf{A}_2 = \mathbf{A}_1 \begin{pmatrix} 1 & \frac{i(\mu+\lambda)\Lambda}{(\mu+\lambda)k} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} i(\mu+\lambda)k\Lambda & -\mu k^2 \\ \mu\Lambda^2 - (2\mu+\lambda)k^2 & \frac{(i(\mu\lambda+2\mu^2)\Lambda^3 - i(2\mu\lambda+3\mu^2))\Lambda k^2}{(\mu+\lambda)k} \end{pmatrix}$$

$$\mathbf{A}_3 = \mathbf{A}_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -\mu k^2 & i(\mu+\lambda)k\Lambda \\ \frac{(i(\mu\lambda+2\mu^2)\Lambda^3 - i(2\mu\lambda+3\mu^2))\Lambda k^2}{(\mu+\lambda)k} & \mu\Lambda^2 - (2\mu+\lambda)k^2 \end{pmatrix}$$

The above situation correspond to case 2

$$\mathbf{A}_4 = \mathbf{A}_3 \begin{pmatrix} 1 & \frac{i(\mu+\lambda)\Lambda}{\mu k} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\mu k^2 & 0 \\ \frac{(i(\mu\lambda+2\mu^2)\Lambda^3 - i(2\mu\lambda+3\mu^2))\Lambda k^2}{(\mu+\lambda)k} & -\frac{(\lambda+2\mu)\Lambda^4 - (2\lambda+4\mu)k^2\Lambda^2 + (\lambda+2\mu)k^4}{k^2} \end{pmatrix}$$

here we are in case 4

$$\mathbf{A}_5 = \begin{pmatrix} 1 & 0 \\ \frac{(i(\mu\lambda+2\mu^2)\Lambda^3 - i(2\mu\lambda+3\mu^2))\Lambda k^2}{\mu(\mu+\lambda)k^3} & 1 \end{pmatrix} \mathbf{A}_4 = \begin{pmatrix} -\mu k^2 & 0 \\ 0 & -\frac{(\lambda+2\mu)(\Lambda^2-k^2)^2}{k^2} \end{pmatrix}$$

finally we obtain ;

The diagonal matrix given by :

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -(\Lambda^2 - k^2)^2 \end{pmatrix}$$

$$E = \begin{pmatrix} -\mu k^2 & 0 \\ \frac{i\mu\Lambda((\lambda+2\mu)\Lambda^2 - (2\lambda+3\mu)k^2)}{(\lambda+\mu)k} & 1 \end{pmatrix}$$

$$F = \begin{pmatrix} \frac{-(\lambda+2\mu)\Lambda^2}{\mu k^2} + 1 & \frac{-i(\lambda+\mu)\Lambda}{\mu k} \\ \frac{i(\lambda+2\mu)^2\Lambda}{(\lambda+\mu)k^3} & \frac{\lambda+2\mu}{k^2} \end{pmatrix}$$

such that $A = EDF$

3 An efficient(optimal) algorithm for the system of linear elasticity

Our goal is to write for the equations of linear elasticity on the whole plane divided into two half-planes an algorithm converging in two iterations. We have shown that the design of an algorithm for the fourth order operator $\mathcal{B} := \Delta^2$ is a key ingredient for this task. Therefore, we derive an algorithm for the operator $\mathcal{B}$ and then, via the Smith factorization, we recast it in a new algorithm for the elasticity system.

We consider the following problem : Find $\phi : R^2 \to R$ such that

$-\triangle^2\phi = f$ in R^2, $|\phi(\overrightarrow{x})| \to 0$ for $|\mathbf{x}| \to \infty$

where f is given right hand side. The domain Ω is decomposed into two halfplanes $\Omega_1 = R^- \times R$ and $\Omega_2 = R^+ \times R$. Let the interface $\{0\} \times R$ be denoted by Γ and $(\mathbf{n}_i)_{i=1,2}$ be the outward normal of $(\Omega_i)_{i=1,2}$. The algorithm, we propose, is given as follows:

Algorithm3.1. We choose the initial values ϕ_1^0 and ϕ_2^0 such that $\phi_1^0 = \phi_2^0$ and $\triangle\phi_1^0 = \triangle\phi_2^0$ on Γ. We obtain $(\phi_i^{n+1})_{i=1,2}$ from $(\phi_i^n)_{i=1,2}$ by the following

iterative procedure:

Correction step. We compute the corrections $(\tilde{\phi}_i^{n+1})_{i=1,2}$ as the solution of the homogeneous local problems

$$\begin{cases} -\triangle^2\tilde{\phi}_i^{n+1} = 0 \, in \, \Omega_i, \\ lim_{|\mathbf{x}|\to 0}|\tilde{\phi}_i^{n+1}| = 0, \\ \frac{\partial\tilde{\phi}_i^{n+1}}{\partial \mathbf{n}_i} = \gamma_1^n \, on \, \Gamma, \\ \frac{\partial\triangle\tilde{\phi}_i^{n+1}}{\partial \mathbf{n}_i} = \gamma_2^n \, on \, \Gamma, \end{cases}$$

where $\gamma_1^n = -\frac{1}{2}(\frac{\partial\phi_1^n}{\partial \mathbf{n}_1} + \frac{\partial\phi_2^n}{\partial \mathbf{n}_2})$ and$\gamma_2^n = -\frac{1}{2}(\frac{\partial\triangle\phi_1^n}{\partial \mathbf{n}_1} + \frac{\partial\triangle\phi_2^n}{\partial \mathbf{n}_2})$.

Udapting step. We update $(\phi_i^{n+1})_{i=1,2}$ by solving the local problems

$$\begin{cases} -\triangle\phi_i^{n+1} = f \, in \, \Omega_i \\ lim_{|\mathbf{x}|\to 0}|\phi_i^{n+1}| = 0, \\ \phi_i^{n+1} = \phi_i^n + \delta_1^{n+1} \, on \, \Gamma \\ \triangle\phi_i^{n+1} = \triangle\phi_i^n + \delta_2^{n+1} \, on \, \Gamma \end{cases}$$

where $\delta_1^{n+1} = \frac{1}{2}(\tilde{\phi}_1^{n+1} + \tilde{\phi}_2^{n+1})$ and $\delta_2^{n+1} = \frac{1}{2}(\triangle\tilde{\phi}_1^{n+1} + \triangle\tilde{\phi}_2^{n+1})$.

Proposition

Algorithm 3.1. converges in two iterations

Proof.

The equations and the algorithm are linear. it suffices to prove convergence to zero of the above algorithm when $f \equiv 0$. We make use of the Fourier transform in the y direction. First of all, as $\phi_1^0 = \phi_2^0$ and $\triangle\phi_1^0 = \triangle\phi_2^0$ on Γ, we obtain the same properties for ϕ_1^1 and ϕ_2^1. Then note that at each step of the algorithm ϕ_i^n satisfies the homogeneous equation in each subdomain

$-\hat{\triangle}\hat{\phi}_i^{\,n} = -(\partial_{xx} - k^2)^2\hat{\phi}_i^{\,n} = 0$

For each k $\in R$, is a fourth order ordinary differential equation in x. The solution in each domain tends to 0 as |x| tends to ∞. We get

$\hat{\phi}_1^n(x,k) = \alpha_1^n(k)e^{|k|x} + \beta_1^n(k)xe^{|k|x}$

$\hat{\phi}_2^n(x,k) = \alpha_2^n(k)e^{-|k|x} + \beta_2^n xe^{-|k|x}$

From $\hat{\phi}_1^1(0,k) = \hat{\phi}_2^1(0,k)$ we have $\alpha_1^1(k) = \alpha_2^1(k)$

From $\hat{\triangle}\hat{\phi}_1^1(0,k) = \hat{\triangle}\hat{\phi}_2^1(0,k)$ we obtain $\beta_1^1(k) = -\beta_2^1(k)$

Therefore we can omit the subscript indicating the number of the subdomain in α and β. Then , we can compute γ_1^1 and γ_2^1 used by the correction step

$\gamma_1^1 = -(|k|\alpha^1(k) + \beta^1(k))$

$\gamma_2^1 = 2k^2\beta^1(k)$

A direct computation shows that the solutions of the correction step $\tilde{\phi}_i^2$, $i=1,2$, are given by :

$\hat{\tilde{\phi}}_1^2(x,k) = -\alpha^1(k)e^{|k|x} - \beta^1(k)xe^{|k|x}$

$\hat{\tilde{\phi}}_2^2(x,k) = -\alpha^1(k)e^{-|k|x} - \beta^1(k)xe^{-|k|x}$

Inserting this into algorithm 2.1. shows that the right hand side of the boundary conditions are zero. Since we assumed $f{\equiv}0$, this shows that $\hat{\phi}_i^2 = 0$ for $i = 1,2$.

From the fourth order operator -$\triangle^2$ to the linear elasticity system.

After having found an optimal algorithm which convergers in two steps for the fourth order operator -$\triangle^2$ problem, we focus on the linear elasticity system . It suffices to replace the operator -$\triangle^2$ by the linear elasticity system in matrix form and ϕ by the last component $(\mathrm{F}(u,v)^T)_2$ of the vector $\mathrm{F}(u,v)^T$ in the boundary conditions. algorithm reads :

Algorithm3.2. We choose the initial values $(u_1^0,v_1^0)$and (u_2^0,v_2^0) such that $(\mathrm{F}(u_1^0,v_1^0)^T)_2 = (\mathrm{F}(u_2^0,v_2^0)^T)_2$ and $\triangle(\mathrm{F}(u_1^0,v_1^0)^T)_2 = \triangle(\mathrm{F}(u_2^0,v_2^0)^T)_2$ on Γ. We compute $((u_i^{n+1},v_i^{n+1}))_{i=1,2}$ from $((u_i^n,v_i^n))_{i=1,2}$ by the following iterative procedure :

Corrrection step. We compute the corrections $((\tilde{u}_i^{n+1},\tilde{v}_i^{n+1}))_{i=1,2}$ as the solution of the homogeneous local problems

$$\left\{\begin{array}{c} S_2(\tilde{u}_i^{n+1},\tilde{v}_i^{n+1}) = 0\, in\, \Omega_i \\ Lim_{|\mathbf{x}|\to\infty}|\tilde{\mathbf{u}}\,_i^{n+1}| = 0, \\ \frac{\partial(F(\tilde{u}_i^{n+1},\tilde{v}_i^{n+1})^T)_2}{\partial \mathbf{n}_i} = \gamma_1^n\, on\, \Gamma, \\ \frac{\partial\triangle(F(\tilde{u}_i^{n+1},\tilde{v}_i^{n+1})^T)_2}{\partial \mathbf{n}_i} = \gamma_2^n\, on\, \Gamma, \end{array}\right.$$

where

$\gamma_1^n{=}-\frac{1}{2}(\frac{\partial(F(u_1^n,v_1^n)^T)_2}{\partial \mathbf{n}_1}+\frac{\partial(F(u_2^n,v_2^n)^T)_2}{\partial \mathbf{n2}})$

$\gamma_2^n{=}-\frac{1}{2}(\frac{\partial\triangle(F(u_1^n,v_1^n)^T)_2}{\partial \mathbf{n}_1}+\frac{\partial\triangle(F(u_2^n,v_2^n)^T)_2}{\partial \mathbf{n}_2})$

Updating step. We update $((u_i^{n+1},v_i^{n+1}))_{i=1,2}$ by solving the local problems:

$$\left\{\begin{array}{c} S_2(u_i^{n+1},v_i^{n+1}) = \mathbf{f}\, in\, \Omega_i, \\ Lim_{|\mathbf{x}|\to\infty}|\mathbf{u}\,_i^{n+1}| = 0, \\ (F(u_i^{n+1},v_i^{n+1})^T)_2 = (F(u_i^n,v_i^n)^T)_2 + \delta_1^{n+1}\, on\, \Gamma \\ \triangle(F(u_i^{n+1},v_i^{n+1})^T)_2 = \triangle(F(u_i^n,v_i^n)^T)_2 + \delta_2^{n+1}\, on\, \Gamma \end{array}\right.$$

where

$\delta_1^{n+1} = \frac{1}{2}[(F(\tilde{u}_1^{n+1},\tilde{v}_1^{n+1})^T)_2 + (F(\tilde{u}_2^{n+1},\tilde{v}_2^{n+1})^T)_2]$

$\delta_2^{n+1}{=}\frac{1}{2}[\triangle(F(\tilde{u}_1^{n+1},\tilde{v}_1^{n+1})^T)_2 + \triangle(F(\tilde{u}_2^{n+1},\tilde{v}_2^{n+1})^T)_2]$

This algorithm seems quite complex since it involves third order derivatives of the unknowns in the boundary conditions on $(\mathrm{F}(\tilde{u}_i,\tilde{v}_i)^T)_2$. Writing $(\mathrm{F}(\tilde{u}_i,\tilde{v}_i)^T)_2 {=}\tilde{u}_i$, it is possible to simplify it. By using the linear elasicity system in the subdomains, we can lower the degree of the derivatives in the boundary conditions. In order to ease the presentation in *Algorithm3.3*. we do not mention that the solutions tend to zero as $|\overrightarrow{x}|\to\infty$.

Algorithm3.3. We choose the initial values (u_1^0,v_1^0) and (u_2^0,v_2^0) such that $v_1^0 = v_2^0$ and $\frac{\partial u_1^0}{\partial \overrightarrow{n_1}} = \frac{\partial u_2^0}{\partial \overrightarrow{n_2}}$ on Γ. We compute $((u_i^{n+1},v_i^{n+1}))_{i=1,2}$ from $((u_i^n,v_i^n))_{i=1,2}$ by the following iterative procedure :

Correction step. We compute the corrections $((\tilde{u}_i^{n+1},\tilde{v}_i^{n+1}))_{i=1,2}$ as the solution of the homogeneous local problems :

$$\left\{\begin{array}{c} S_2(\tilde{u}_1^{n+1},\tilde{v}_1^{n+1}) = 0\, in\, \Omega_1, \\ \frac{\partial\tilde{u}_1^{n+1}}{\partial x} = -\frac{1}{2}(\frac{\partial u_1^n}{\partial x} - \frac{\partial u_2^n}{\partial x})\, on\, \Gamma, \\ \frac{\partial\tilde{u}_1^{n+1}}{\partial x} + \frac{\partial\tilde{v}_1^{n+1}}{\partial y} = \gamma_{2,1}^n\, on\, \Gamma \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{c} S_2(\tilde{u}_2^{n+1},\tilde{v}_2^{n+1}) = 0\, in\, \Omega_2, \\ \frac{\partial\tilde{u}_2^{n+1}}{\partial x} = \frac{1}{2}(\frac{\partial u_1^n}{\partial x} - \frac{\partial u_2^n}{\partial x})\, on\, \Gamma, \\ -\frac{\partial\tilde{u}_2^{n+1}}{\partial x} - \frac{\partial\tilde{v}_2^{n+1}}{\partial y} = \gamma_{2,1}^n\, on\, \Gamma \end{array}\right.$$

where

$\gamma_{2,1}^n = -\frac{1}{2}(\frac{\partial u_1^n}{\partial x} + \frac{\partial v_1^n}{\partial y} - \frac{\partial u_2^n}{\partial x} - \frac{\partial v_2^n}{\partial y})$

Updating step. We update $((u_i^{n+1}, v_i^{n+1}))_{i=1,2}$ by solving the local problems

$$\begin{cases} S_2(u_i^{n+1}, v_i^{n+1}) = \overrightarrow{f}\, in\, \Omega_{i,} \\ u_i^{n+1} = u_i^n + \frac{1}{2}(\tilde{u}_1^{n+1} + \tilde{u}_2^{n+1})\, on\, \Gamma \\ \frac{\partial u_i^{n+1}}{\partial y} - \frac{\partial v_i^{n+1}}{\partial x} = \frac{\partial u_i^n}{\partial y} - \frac{\partial v_i^n}{\partial x} + \delta_{2,1}^n\, on\, \Gamma \end{cases}$$

where
$\delta_{2,1}^n = \frac{1}{2}(\frac{\partial \tilde{u}_1^{n+1}}{\partial y} - \frac{\partial \tilde{v}_1^{n+1}}{\partial x} + \frac{\partial \tilde{u}_2^{n+1}}{\partial y} - \frac{\partial \tilde{v}_2^{n+1}}{\partial x})$

Schwarz Overlap Scheme applied to the linear elasticity system

We want to solve

$$S_2(\mathbf{w}) = \overrightarrow{f}\, in\, \Omega_1 \bigcup \Omega_2$$

where $\mathbf{w} = (u, v)$.
The schwarz algorithm runs like this :
Start from (u_1^0, v_1^0), (u_2^0, v_2^0) we compute $\mathbf{w}_1^{n+1}$, $\mathbf{w}_2^{n+1}$ from $\mathbf{w}_1^n$, $\mathbf{w}_2^n$as follows :

$$\begin{cases} S_2(\mathbf{w}_1^{n+1}) = f\, in\, \Omega_1 \\ \mathbf{w}_1^{n+1} = \, on\, \partial\Omega_1 \bigcap \Omega_2 \end{cases}$$

and

$$\begin{cases} S_2(\mathbf{w}_2^{n+1}) = f\, in\, \Omega_2 \\ \mathbf{w}_2^{n+1} = \mathbf{w}_1^n, on\, \partial\Omega_2 \bigcap \Omega_1 \end{cases}$$

Here we take 1 and 2 to be rectangle, we apply the algorithm starting from zero.
Figure : The 2 overlapping mesh TH and th

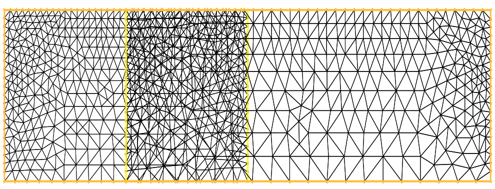


Figure :Displacement fields during the iterations

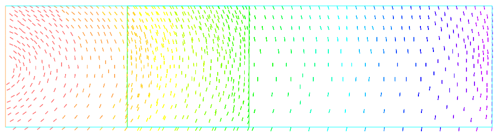


Figure : Final configuration of the beam after convergence