4. THE CENTER MANIFOLD METHOD

- **Existence of an invariant manifold**

- Linear systems

The state-space $\mathcal{X} \subseteq \mathbb{R}^N$ of the linear system $\dot{\mathbf{x}}(t) = \mathbf{J}\mathbf{x}(t)$ is direct sum of three invariant sub-spaces, i.e. $\mathcal{X} = \mathcal{X}_c \oplus \mathcal{X}_s \oplus \mathcal{X}_u$, where:

- $\mathcal{X}_c$ is the *center subspace*, of dimension N_c, spanned by the (generalized) eigenvectors associated with non-hyperbolic eigenvalues of $\mathbf{J}$;
- $\mathcal{X}_s$ is the *stable subspace*, of dimension N_s, spanned by the (generalized) eigenvectors associated with hyperbolic eigenvalues of $\mathbf{J}$ having negative real part;
- $\mathcal{X}_u$ is the *unstable subspace*, of dimension N_u, spanned by the (generalized) eigenvectors associated with non-hyperbolic eigenvalues of $\mathbf{J}$ having positive real part.

- Nonlinear systems

We consider the nonlinear system (in local form):

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t), \boldsymbol{\mu})$$

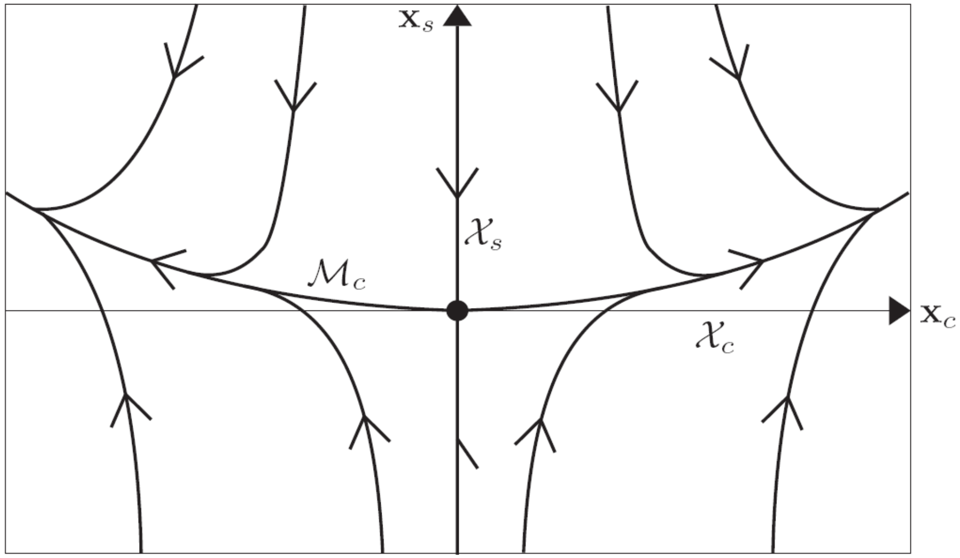
admitting the critical equilibrium $(\mathbf{x}=\mathbf{0}, \boldsymbol{\mu}_c=\mathbf{0})$. We assume that $\mathbf{J} := \mathbf{F}_{\mathbf{x}}(\mathbf{0},\mathbf{0})$ posses N_c >0 critical eigenvalues, N_s >0 stable eigenvalues and N_u =0 unstable eigenvalues.

The *Center Manifold Theorem* states that the asymptotic dynamics of the system around the equilibrium point $\mathbf{x}=\mathbf{0}$, at the critical value of the parameters $\boldsymbol{\mu}=\boldsymbol{\mu}_c$, takes place on a (critical) manifold $\mathcal{M}_c \in \mathcal{X}$, which has the following properties:

- $\mathcal{M}_c$ has dimension N_c;
- $\mathcal{M}_c$ is *tangent* to the critical subspace $\mathcal{X}_c$ at $\mathbf{x}=\mathbf{0}$;
- $\mathcal{M}_c$ is *attractive*, i.e. all the orbits tend to it when $t\to\infty$.

The center manifold is therefore an N_c-dimensional surface in the $N=N_c+N_s$-dimensional state-space.

- Example:



- To analyze the asymptotic dynamics, it needs:

  (a) to find the center manifold $\mathcal{M}_c$;

  (b) to obtain the *reduced* N_c –dimensional equations governing the motion on $\mathcal{M}_c$ (*bifurcation equations*).

- **Dependence of the CM on parameters**

Since we are interested not only in the dynamics at $\boldsymbol{\mu} = \boldsymbol{\mu}_c$, but also at the dynamics at $\boldsymbol{\mu}$ close to $\boldsymbol{\mu}_c$, we can use the 'trick' to consider $\boldsymbol{\mu}$ as additional 'critical' variables, by considering the *extended dynamical system*:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t), \boldsymbol{\mu}(t)) \\ \dot{\boldsymbol{\mu}}(t) = \mathbf{0} \end{cases}$$

Therefore, the critical subspace becomes $\mathcal{X}_c^+ := \mathcal{X}_c \oplus \mathcal{P}$ with $\mathcal{P} := \{\boldsymbol{\mu}\}$ the parameter space. Hence:

- $\mathcal{M}_c^+$ has dimension N_c+M;
- $\mathcal{M}_c^+$ is *tangent* to the critical subspace $\mathcal{X}_c^+$;
- $\mathcal{M}_c^+$ is *attractive*, i.e. all the orbits tend to it when $t \to \infty$.

- **Reduction process**

  - Equations of motion, expanded:

    By expanding $\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t), \boldsymbol{\mu})$ (and ignoring the dummy equations $\dot{\boldsymbol{\mu}} = \mathbf{0}$) for small $\mathbf{x}(t)$ and $\boldsymbol{\mu}$ close to $\boldsymbol{\mu}_c$, we have:

$$\dot{\mathbf{x}}(t) = \mathbf{J}\mathbf{x}(t) + \mathbf{f}(\mathbf{x}(t), \boldsymbol{\mu}), \quad |\mathbf{f}| = \mathrm{O}(|\mathbf{x}(t)|^2, |\boldsymbol{\mu}|^2)$$

  - Linear transformation of the variables:

    After letting $\mathbf{x}(t) := (\mathbf{x}_c(t), \mathbf{x}_s(t))$, with $\mathbf{x}_c(t) \in \mathcal{X}_c$ *critical variables* and $\mathbf{x}_s(t) \in \mathcal{X}_s$ *stable variables*, a proper linear transformation uncouples the linear part of the equations (t omitted):

$$\begin{pmatrix} \dot{\mathbf{x}}_c \\ \dot{\mathbf{x}}_s \end{pmatrix} = \begin{pmatrix} \mathbf{J}_c & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_s \end{pmatrix} \begin{pmatrix} \mathbf{x}_c \\ \mathbf{x}_s \end{pmatrix} + \begin{pmatrix} \mathbf{f}_c(\mathbf{x}_c, \mathbf{x}_s, \boldsymbol{\mu}) \\ \mathbf{f}_s(\mathbf{x}_c, \mathbf{x}_s, \boldsymbol{\mu}) \end{pmatrix}$$

➢ Center manifold: *active and passive* coordinates

Cartesian equations for the CM:

$$\mathbf{x}_s = \mathbf{h}(\mathbf{x}_c, \boldsymbol{\mu})$$

Tangency requirements:

$$\mathbf{h}(\mathbf{0},\mathbf{0}) = \mathbf{0}, \quad \mathbf{h}_{\mathbf{x}}(\mathbf{0},\mathbf{0}) = \mathbf{0}, \quad \mathbf{h}_{\boldsymbol{\mu}}(\mathbf{0},\mathbf{0}) = \mathbf{0}\ .$$

The equation expresses the stable variables $\mathbf{x}_s$ as (unknown) functions of the critical variables $\mathbf{x}_c$; therefore $\mathbf{x}_c$ are also said *active coordinates*, and $\mathbf{x}_s$ *passive coordinates*.

➢ Time derivative of $\mathbf{x}_s$:

The passive character of $\mathbf{x}_s$ also holds for time-derivatives. By using the chain rule and the upper partition of the equations of motion:

$$\begin{aligned}\dot{\mathbf{x}}_s &= \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{h}(\mathbf{x}_c, \boldsymbol{\mu}) = \mathbf{h}_{\mathbf{x}}(\mathbf{x}_c, \boldsymbol{\mu})\dot{\mathbf{x}}_c \\ &= \mathbf{h}_{\mathbf{x}}(\mathbf{x}_c, \boldsymbol{\mu})(\mathbf{J}_c\mathbf{x}_c + \mathbf{f}_c(\mathbf{x}_c, \mathbf{h}(\mathbf{x}_c), \boldsymbol{\mu}))\end{aligned}$$

- Equation for the center manifold:

  The lower-partition of the equation of motion supplies the equation for determining $\mathcal{M}_c$:

$$\mathbf{h_x}(\mathbf{x}_c,\boldsymbol{\mu})(\mathbf{J}_c\mathbf{x}_c+\mathbf{f}_c(\mathbf{x}_c,\mathbf{h}(\mathbf{x}_c),\boldsymbol{\mu}))=\mathbf{J}_s\mathbf{h}(\mathbf{x}_c)+\mathbf{f}_s(\mathbf{x}_c,\mathbf{h}(\mathbf{x}_c),\boldsymbol{\mu})$$

  This equation can be solved, e.g., by using power series expansions, (starting from degree-2 polynomial, due to the required tangency):

$$\mathbf{h}(\mathbf{x}_c,\boldsymbol{\mu})=\boldsymbol{\alpha}_2\mathbf{z}^2+\boldsymbol{\alpha}_3\mathbf{z}^3+\cdots,\qquad \mathbf{z}:=\{\mathbf{x}_c,\boldsymbol{\mu}\}$$

- Bifurcation equations:

  The upper-partition governs the dynamics on $\mathcal{M}_c$:

$$\dot{\mathbf{x}}_c=\mathbf{J}_c\mathbf{x}_c+\mathbf{f}_c(\mathbf{x}_c,\mathbf{h}(\mathbf{x}_c),\boldsymbol{\mu})$$

- **Example 1: a static bifurcation**

  - Nonlinear, 2-dimensional system:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} \mu & 0 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} xy + cx^3 \\ bx^2 \end{pmatrix}$$

  - Critical point, critical and stable coordinates:

$$\mu_c = 0, \quad \mathbf{x}_c = \{x\}, \quad \mathbf{x}_s = \{y\}$$

  - Equation for the Center Manifold:

$$y = h(x,\mu) \quad \Rightarrow \quad h_x(x,\mu)[\mu x + xh(x,\mu) + cx^3] = -h(x,\mu) + bx^2$$

  - Power series expansion:

$$y = h(x,\mu) = \{\alpha_{20}x^2 + \alpha_{11}x\mu + \alpha_{02}\mu^2\} + \{\alpha_{30}x^3 + \cdots\} + \cdots$$

➢ Zeroing separately the different powers:

$$x^2 \; : -\alpha_{20} + b = 0$$
$$x\mu : -\alpha_{11} = 0$$
$$\mu^2 : -\alpha_{02} = 0$$
$$\cdots\cdots\cdots$$

➢ By solving for α's coefficients, at the lowest order, we have:

$$y = bx^2 + \mathrm{O}(\left|(x,\mu)\right|^3)$$

➢ Bifurcation equation:

$$\dot{x} = \mu x + (b+c)x^3$$

If $b+c<0$, a supercritical (stable) pitchfork occurs; if $b+c>0$, a sub-critical (unstable) pitchfork occurs; if $b+c=0$, higher-order terms must be evaluated.

- **Example 2: a dynamical bifurcation**

  - 3-dimensional nonlinear system:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \mu & -1 & 0 \\ 1 & \mu & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} xz \\ yz \\ x^2 + kxy + y^2 \end{pmatrix}$$

  - Equation for CM:

$$z = \alpha_1 x^2 + \alpha_2 y^2 + \alpha_3 xy + \alpha_4 \mu x + \alpha_5 \mu y + \mathrm{O}(3)$$

  - z-equation (passive coordinate), transformed:

$$\begin{aligned} &2(\alpha_2 - \alpha_1)xy + \alpha_3(x^2 - y^2) + \alpha_5 \mu x - \alpha_4 \mu y \\ &= -(\alpha_1 x^2 + \alpha_2 y^2 + \alpha_3 xy + \alpha_4 \mu x + \alpha_5 \mu y) + x^2 + kxy + y^2 + \mathrm{O}(3) \end{aligned}$$

➢ Zeroing separately the different powers in the z-equation:

$$\begin{cases} x^2: & \alpha_3+\alpha_1-1=0 \\ y^2: & -\alpha_3+\alpha_2-1=0 \\ xy: & 2(\alpha_2-\alpha_1)+\alpha_3-k=0 \\ \mu x: & \alpha_4+\alpha_5=0 \\ \mu y: & \alpha_5-\alpha_4=0 \end{cases} \Rightarrow \begin{cases} \alpha_1=1-k/5 \\ \alpha_2=1+k/5 \\ \alpha_3=k/5 \\ \alpha_4=0 \\ \alpha_5=0 \end{cases}$$

➢ Bifurcation equations:

$$\begin{cases} \dot{x}=\mu x-y+x[(1-k/5)x^2+k/5xy+(1+k/5)y^2] \\ \dot{y}=\mu y-x+y[(1-k/5)x^2+k/5xy+(1+k/5)y^2] \end{cases}$$

Note: if $k\neq 0$, the inertia $\dot{z}$ contributes to the motion; if $k=0$, z contributes statically.

6.THE NORMAL FORM THEORY

- **Scope:**
  To use a smooth *nonlinear coordinate transformation*, in order to put the bifurcation equation in the simplest form.

- **Algorithm:**

Equations of motion: $\dot{\mathbf{x}} = \mathbf{J}\mathbf{x} + \mathbf{f}(\mathbf{x})$

Transformed Equations: $\dot{\mathbf{y}} = \mathbf{J}\mathbf{y} + \mathbf{g}(\mathbf{y})$

Near-identity transformation: $\mathbf{x} = \mathbf{y} + \mathbf{h}(\mathbf{y})$

where $\mathbf{x}$ (possibly) includes the dummy variables $\boldsymbol{\mu}$.

- Transformed equation in the $\mathbf{h}(\mathbf{y})$ unknown:

  By differentiating the nearly-identity transformation:

$$\dot{\mathbf{x}} = [\mathbf{I} + \mathbf{h}_{\mathbf{y}}(\mathbf{y})]\dot{\mathbf{y}}$$

  the equation of motion is transformed into:

$$[\mathbf{I} + \mathbf{h}_{\mathbf{y}}(\mathbf{y})][\mathbf{J}\mathbf{y} + \mathbf{g}(\mathbf{y})] = \mathbf{J}[\mathbf{y} + \mathbf{h}(\mathbf{y})] + \mathbf{f}(\mathbf{y} + \mathbf{h}(\mathbf{y}))$$

  or:

$$\mathbf{h}_{\mathbf{y}}(\mathbf{y})\mathbf{J}\mathbf{y} - \mathbf{J}\mathbf{h}(\mathbf{y}) = \mathbf{f}(\mathbf{y} + \mathbf{h}(\mathbf{y})) - [\mathbf{I} + \mathbf{h}_{\mathbf{y}}(\mathbf{y})]\mathbf{g}(\mathbf{y})$$

  which is a differential equation for $\mathbf{h}(\mathbf{y})$ for any given $\mathbf{g}(\mathbf{y})$. A series solution is often necessary.

- Series solution:

$$\begin{pmatrix} \mathbf{f}(\mathbf{y}) \\ \mathbf{g}(\mathbf{y}) \\ \mathbf{h}(\mathbf{y}) \end{pmatrix} = \begin{pmatrix} \mathbf{f}_2(\mathbf{y}) \\ \mathbf{g}_2(\mathbf{y}) \\ \mathbf{h}_2(\mathbf{y}) \end{pmatrix} + \begin{pmatrix} \mathbf{f}_3(\mathbf{y}) \\ \mathbf{g}_3(\mathbf{y}) \\ \mathbf{h}_3(\mathbf{y}) \end{pmatrix} + \cdots$$

- Chain of equations:

$$\mathbf{h}_{2,\mathbf{y}}(\mathbf{y})\mathbf{J}\mathbf{y} - \mathbf{J}\mathbf{h}_2(\mathbf{y}) = \mathbf{f}_2(\mathbf{y}) - \mathbf{g}_2(\mathbf{y})$$
$$\mathbf{h}_{3,\mathbf{y}}(\mathbf{y})\mathbf{J}\mathbf{y} - \mathbf{J}\mathbf{h}_3(\mathbf{y}) = \mathbf{f}_3(\mathbf{y}) + \mathbf{f}_{2,\mathbf{y}}(\mathbf{y})\mathbf{h}_2(\mathbf{y}) - \mathbf{h}_{2,\mathbf{y}}(\mathbf{y})\mathbf{g}_2(\mathbf{y}) - \mathbf{g}_3(\mathbf{y})$$
$$\cdots\cdots\cdots$$

i.e.:

$$\mathcal{L}\mathbf{h}_k(\mathbf{y}) = \hat{\mathbf{f}}_k(\mathbf{y}) - \mathbf{g}_k(\mathbf{y}), \quad k = 2, 3, \cdots$$

- Homogeneous polynomial of *k*-degree:

$$\hat{\mathbf{f}}_k(\mathbf{y}) = \sum_{m=1}^{M_k} \alpha_{km}\mathbf{p}_{km}(\mathbf{y}), \quad \mathbf{g}_k(\mathbf{y}) = \sum_{m=1}^{M_k} \beta_{km}\mathbf{p}_{km}(\mathbf{y}), \quad \mathbf{h}_k(\mathbf{y}) = \sum_{m=1}^{M_k} \gamma_{km}\mathbf{p}_{km}(\mathbf{y})$$

- Zeroing the independent monomials:

$$\mathbf{L}_k\boldsymbol{\gamma}_k = \boldsymbol{\alpha}_k - \boldsymbol{\beta}_k$$

where (if **J** is diagonal):

$$\mathbf{L}_k = \mathrm{diag}[\Lambda_i], \qquad \Lambda_i := \sum_{j=1}^{N}(m_j\lambda_j - \lambda_i), \quad \sum_{j=1}^{N} m_j = k$$

- Resonance:

Since $\lambda_j = 0, 0, \cdots, \pm i\omega_1, \pm i\omega_2, \cdots,$ $\Lambda_i = 0$ for some i (i.e. $\mathbf{L}_k$ is singular); hence, $\boldsymbol{\beta}_k \neq \mathbf{0}$ must be taken for compatibility, and *resonant terms survive in the normal form!!!*

- **Example 1: System independent of parameters**

  - System *at* a Hopf bifurcation:

$$\begin{pmatrix} \dot{x} \\ \dot{\overline{x}} \end{pmatrix} = \begin{bmatrix} i\omega & 0 \\ 0 & -i\omega \end{bmatrix} \begin{pmatrix} x \\ \overline{x} \end{pmatrix} + \begin{pmatrix} \alpha_{31} x^3 + \alpha_{32} x^2 \overline{x} + \alpha_{33} x \overline{x}^2 + \alpha_{34} \overline{x}^3 \\ \overline{\alpha}_{31} \overline{x}^3 + \overline{\alpha}_{32} \overline{x}^2 x + \overline{\alpha}_{33} \overline{x} x^2 + \overline{\alpha}_{34} x^3 \end{pmatrix}, \quad x, \alpha \in \mathbb{C}$$

  - Normal form:

$$\dot{y} = i\omega y + \beta_{31} y^3 + \beta_{32} y^2 \overline{y} + \beta_{33} y \overline{y}^2 + \beta_{34} \overline{y}^3$$

  - Near-identity transformation:

$$x = y + \gamma_{31} y^3 + \gamma_{32} y^2 \overline{y} + \gamma_{33} y \overline{y}^2 + \gamma_{34} \overline{y}^3$$

○ Equation for the coefficients:

$$\begin{bmatrix} 2i\omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2i\omega & 0 \\ 0 & 0 & 0 & -4i\omega \end{bmatrix} \begin{pmatrix} \gamma_{31} \\ \gamma_{32} \\ \gamma_{33} \\ \gamma_{34} \end{pmatrix} = \begin{pmatrix} \alpha_{31} \\ \alpha_{32} \\ \alpha_{33} \\ \alpha_{34} \end{pmatrix} - \begin{pmatrix} \beta_{31} \\ \beta_{32} \\ \beta_{33} \\ \beta_{34} \end{pmatrix}$$

○ Solution:

$$\beta_{31} = \beta_{33} = \beta_{34} = 0, \quad \beta_{32} = \alpha_{32}$$

○ Normal form:

$$\dot{y} = i\omega y + \alpha_{32} y^2 \overline{y}$$

The term $y^2\overline{y}$ is *resonant*, since $2\lambda_1 + \lambda_2 = \lambda_1$, i.e. $2(i\omega) + (-i\omega) = i\omega$

- **Example 2: System dependent on a parameter**

  - System *close* to a Hopf bifurcation:

$$\begin{pmatrix} \dot{x} \\ \dot{\overline{x}} \end{pmatrix} = \begin{bmatrix} \mu + i\omega & 0 \\ 0 & \mu - i\omega \end{bmatrix} \begin{pmatrix} x \\ \overline{x} \end{pmatrix} + \begin{pmatrix} \alpha_{31} x^3 + \alpha_{32} x^2 \overline{x} + \alpha_{33} x \overline{x}^2 + \alpha_{34} \overline{x}^3 \\ \overline{\alpha}_{31} \overline{x}^3 + \overline{\alpha}_{32} \overline{x}^2 x + \overline{\alpha}_{33} \overline{x} x^2 + \overline{\alpha}_{34} x^3 \end{pmatrix}, \quad x, \alpha \in \mathbb{C}$$

or, extending the state-space:

$$\begin{pmatrix} \dot{x} \\ \dot{\overline{x}} \\ \dot{\mu} \end{pmatrix} = \begin{bmatrix} i\omega & 0 & 0 \\ 0 & -i\omega & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ \overline{x} \\ \mu \end{pmatrix} + \underbrace{\begin{pmatrix} \mu x \\ \mu \overline{x} \\ 0 \end{pmatrix}}_{\mathrm{O}(2)} + \underbrace{\begin{pmatrix} \alpha_{31} x^3 + \alpha_{32} x^2 \overline{x} + \alpha_{33} x \overline{x}^2 + \alpha_{34} \overline{x}^3 \\ \overline{\alpha}_{31} \overline{x}^3 + \overline{\alpha}_{32} \overline{x}^2 x + \overline{\alpha}_{33} \overline{x} x^2 + \overline{\alpha}_{34} x^3 \\ 0 \end{pmatrix}}_{\mathrm{O}(3)}$$

○ First equation:

$$\dot{x} = i\omega x + (\mu x) + (\alpha_{31}x^3 + \alpha_{32}x^2\overline{x} + \alpha_{33}x\overline{x}^2 + \alpha_{34}\overline{x}^3)$$

○ Normal form (reduced number of monomials tried):

$$\dot{y} = i\omega y + \beta_{31}y^3 + \beta_{32}y^2\overline{y} + \beta_{33}y\overline{y}^2 + \beta_{34}\overline{y}^3$$
$$+(\beta_{20}\mu y) + \mu(\beta_{35}y^2 + \beta_{36}y\overline{y} + \beta_{37}\overline{y}^2) + \text{ignored terms}$$

○ Near-identity transformation (reduced number of monomials tried):

$$x = y + \gamma_{31}y^3 + \gamma_{32}y^2\overline{y} + \gamma_{33}y\overline{y}^2 + \gamma_{34}\overline{y}^3$$
$$+(\gamma_{20}\mu y) + \mu(\gamma_{35}y^2 + \gamma_{36}y\overline{y} + \gamma_{37}\overline{y}^2) + \text{ignored terms}$$

○ By operating as in Example 1, at O(2) it follows:

$$0\,\gamma_{20} = 1 - \beta_{20} \quad \Rightarrow \quad \beta_{20} = 1,\, \gamma_{20} = 0$$

○ At O(3), in addition to the equations of Example 1, one has:

$$\begin{bmatrix} i\omega & 0 & 0 \\ 0 & -i\omega & 0 \\ 0 & 0 & -3i\omega \end{bmatrix} \begin{pmatrix} \gamma_{35} \\ \gamma_{36} \\ \gamma_{37} \end{pmatrix} = \frac{i}{3\omega} \begin{pmatrix} 3\alpha_{21} \\ 3\alpha_{22} \\ \alpha_{23} \end{pmatrix} - \begin{pmatrix} \beta_{35} \\ \beta_{36} \\ \beta_{37} \end{pmatrix}$$

Since no further resonances appear,

$$\beta_{35} = \beta_{36} = \beta_{37} = 0\,.$$

○ Normal form:

$$\dot{y} = (\mu + i\omega)\, y + \alpha_{32}\, y^2 \bar{y}$$

○ <u>Amplitude equation</u>:

Changing the variables according to:

$$y(t) = A(t)\,\mathrm{e}^{i\omega t}, \quad A(t) \in \mathbb{C}$$

the normal form is transformed into:

$$[\dot{A}(t) + i\omega A(t)]\,\mathrm{e}^{i\omega t} = (\mu + i\omega)A(t)\,\mathrm{e}^{i\omega t} + \alpha_{32} A^2(t)\bar{A}(t)\,\mathrm{e}^{(2-1)i\omega t}$$

or:

$$\dot{A}(t) = \mu A(t) + \alpha_{32} A^2(t)\bar{A}(t)$$

This is called *Amplitude Modulation Equation* (AME).

Since $\dot{A} = \mathrm{O}(\mu A, A^3)$, the AME describes a <u>slow modulation.</u> Therefore, the change of variable <u>filters the fast dynamics</u>.

○ In polar form:

$$A(t) := \frac{1}{2} a(t) \exp[i\theta(t)]$$

hence:

$$\frac{1}{2}[\dot{a}(t) + a\dot{\theta}(t))]\cancel{e^{i\theta(t)}} = \frac{1}{2}\mu a(t)\cancel{e^{i\theta(t)}} + \frac{1}{8}\alpha_{32} a(t)^3 \cancel{e^{(2-1)i\theta(t)}}$$

and the real normal form reads (*amplitude-phase modulation equations*):

$$\begin{cases} \dot{a} = \mu a + \frac{1}{4}\mathrm{Re}[\alpha_{32}]a^3 \\ \dot{\theta} = \frac{1}{4}\mathrm{Im}[\alpha_{32}]a^2 \end{cases}$$

- **Example 3: Non-diagonalizable system**
  - Double-zero bifurcation equations:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \alpha_{21} x_1^2 + \alpha_{22} x_1 x_2 + \alpha_{23} x_2^2 \\ \alpha_{24} x_1^2 + \alpha_{25} x_1 x_2 + \alpha_{26} x_2^2 \end{pmatrix}$$

  - Normal Form:

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} \beta_{21} y_1^2 + \beta_{22} y_1 y_2 + \beta_{23} y_2^2 \\ \beta_{24} y_1^2 + \beta_{25} y_1 y_2 + \beta_{26} y_2^2 \end{pmatrix}$$

  - Near-Identity transformation:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} \gamma_{21} y_1^2 + \gamma_{22} y_1 y_2 + \gamma_{23} y_2^2 \\ \gamma_{24} y_1^2 + \gamma_{25} y_1 y_2 + \gamma_{26} y_2^2 \end{pmatrix} + \cdots$$

○ By zeroing the coefficients of the three monomials in the two transformed equations:

$$\left[\begin{array}{ccc:ccc} 0 & 0 & 0 & -1 & 0 & 0 \\ 2 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{pmatrix} \gamma_{21} \\ \gamma_{22} \\ \gamma_{23} \\ \hdashline \gamma_{24} \\ \gamma_{25} \\ \gamma_{26} \end{pmatrix} = \begin{pmatrix} \alpha_{21}-\beta_{21} \\ \alpha_{22}-\beta_{22} \\ \alpha_{23}-\beta_{23} \\ \hdashline \alpha_{24}-\beta_{24} \\ \alpha_{25}-\beta_{25} \\ \alpha_{26}-\beta_{26} \end{pmatrix}$$

$$\text{Since}\,\text{Rank}[\mathbf{L}_2] = 4:$$

$$\mathcal{K}(\mathbf{L}_2) = \text{span}\{\mathbf{u}_{21}, \mathbf{u}_{22}\}, \quad \mathbf{u}_{21} := (0,1,0,0,0,1)^T, \mathbf{u}_{22} = (0,0,1,0,0,0)^T$$

$$\mathcal{K}(\mathbf{L}_2^T) = \text{span}\{\mathbf{v}_{21}, \mathbf{v}_{22}\}, \quad \mathbf{v}_{21} = (2,0,0,0,1,0)^T, \mathbf{v}_{22} = (0,0,0,1,0,0)^T$$

○ Compatibility condition:

$$2(\alpha_{21}-\beta_{21})+(\alpha_{25}-\beta_{25})=0, \qquad \alpha_{24}-\beta_{24}=0$$

It is possible to take $\beta_{22}=\beta_{23}=\beta_{26}=0$ and $\beta_{21}=0$ *or* $\beta_{25}=0$.

○ By taking $\beta_{25}=0, \beta_{21}=\alpha_{21}+\alpha_{25}/2$ the NF reads (*Takens normal form*):

$$\begin{pmatrix}\dot{y}_1\\ \dot{y}_2\end{pmatrix}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}+\begin{pmatrix}(\alpha_{21}+\frac{1}{2}\alpha_{25})y_1^2\\ \alpha_{24}y_1^2\end{pmatrix}$$

○ By taking $\beta_{21}=0, \beta_{25}=\alpha_{25}+2\alpha_{21}$ the NF reads (*Bogdanov normal form)*:

$$\begin{pmatrix}\dot{y}_1\\ \dot{y}_2\end{pmatrix}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}+\begin{pmatrix}0\\ \alpha_{24}y_1^2+(\alpha_{25}+2\alpha_{21})y_1y_2\end{pmatrix}$$

<u>The Normal Form is not unique</u>!!