Chapter 3

Laplace Equation

Undoubtedly, the *Laplace Equation* is among the most important PDEs. Traditionally, we call

$$-\Delta u(x) = 0, \qquad x \in \Omega \subset \mathbb{R}^n, \quad n \geq 2. \tag{3.1}$$

the *Laplace Equation*, and its non-homogeneous companion

$$-\Delta u(x) = f(x), \qquad x \in \Omega \subset \mathbb{R}^n, \quad n \geq 2, \tag{3.2}$$

the *Poisson Equation.*

We recall that the *Laplacian* of u is

$$\Delta u = \sum_{i-1}^{n} u_{x_i x_i}.$$

In both equations, $x = (x_1, \cdots, x_n) \in \Omega \subset \mathbb{R}^n$ and the unknown is $u : \bar{\Omega} \to \mathbb{R}$. In (3.2), the function $f : \Omega \to \mathbb{R}$ is given fucntion. For later use, we have the following defintion:

Definition 3.0.1 *If $u \in C^2(\Omega)$ satisfies (3.1) in Ω, we call it a harmonic function. $u \in C^2(\Omega)$ is called sub-harmonic (or super-harmonic) if it satisfies*

$$-\Delta u \leq 0 (\geq 0).$$

Laplace equation and Poisson equations model many static physical fields with and without sources. They appear in the modeling of gravity, electric force, and the chemical concentration in equilibrium, and much more. Let V be any smooth subregion in Ω, the net flux of u through ∂V is zero:

$$\int_{\partial V} F \bullet \nu \, dS = 0,$$

where F is the flux density and ν the unit outer normal of ∂V. By Divergence Theorem (see Theorem 3.1.1 below), we have

$$\int_V \nabla \bullet F \, dx = \int_{\partial V} F \bullet \nu \, dS = 0,$$

and so

$$\nabla \bullet F = 0, \; in \; \Omega.$$

In many occasions, the flux F is proportional to the gradient Du, pointing in the opposite direction. Therefore, one has

$$F = -aDu, \; a > 0. \tag{3.3}$$

If u stands for the

$$\begin{cases} \text{chemical concentration} \\ \text{temperature} \\ \text{electrostatic potential,} \end{cases}$$

the equation (3.3) is

$$\begin{cases} \text{Fick's law of diffusion} \\ \text{Fourier's law of heat conduction} \\ \text{Ohm's law of electrical conduction.} \end{cases}$$

Laplace equation also arises in the study of analytic functions and the probabilistic investigation of Brownian motion.

3.1 The fundamental solution

From the form of Laplace equation, we see that all directions have the same weights in the equation. Namely, the equation is invariant under any orthogonal transformation of coordinates. We thus seek to solve the equation explicitly by looking for the *radial* solution which has the form

$$u(x) = v(r), \qquad r = |x|,$$

where v is to be determined so that (3.1) holds. For $i \in \{1, \ldots, n\}$,

$$u_{x_i} = v'(r) r_{x_i} = v'(r) \frac{x_i}{r}, \quad u_{x_i x_i} = v''(r) \frac{x_i^2}{r^2} + v'(r) \left(\frac{1}{r} - \frac{x_i^2}{r^3} \right).$$

One has

$$\Delta u = v''(r) + \frac{n-1}{r} v'(r) = 0.$$

If $v' \neq 0$, we deduce

$$(\log(v'))' = \frac{v''}{v'} = \frac{1-n}{r}$$

and hence

$$v'(r) = \frac{a}{r^{n-1}},$$

for some constant a. Therefore, if $r > 0$, we find

$$v(r) = \begin{cases} b \log r + c & \text{for } \ n = 2 \\ \dfrac{b}{r^{n-2}} + c & \text{for } \ n \geq 3, \end{cases}$$

where a and b are constants. We therefore discover the *fundamental solution* of Laplace equation:

$$\Phi(x) = \begin{cases} -\dfrac{1}{2\pi} \log |x| & \text{for } \ n = 2 \\ \dfrac{1}{n(n-2)\alpha(n)} \dfrac{1}{|x|^{n-2}} & \text{for } \ n \geq 3, \end{cases} \tag{3.4}$$

where

$$\alpha(n) = \frac{2\pi^{n/2}}{n\Gamma(n/2)}$$

is the volume of the unite ball in $\mathbb{R}^n$. The reason for particular choices of the constants will be explained later.

From the derivation, we also have the following estimates

$$|D\Phi(x)| \leq \frac{C}{|x|^{n-1}}, \ |D^2\Phi(x)| \leq \frac{C}{|x|^n}, \ (x \neq 0), \tag{3.5}$$

for some constant $C > 0$.

3.1.1 Green's formula

In this subsection, we summarize some useful formulas due to Green. These formulas are convenient in the computations related to Laplacian. These are derived from the Divergence Theorem (or, Gauss formula):

Theorem 3.1.1 *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let ν be the unit outer normal of $\partial\Omega$. For any smooth vector field $w \in C^1(\bar{\Omega})$, it holds that*

$$\int_\Omega \nabla \bullet w \ dx = \int_{\partial\Omega} w \bullet \nu \ dS. \tag{3.6}$$

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let ν be the unit outer normal of $\partial\Omega$. For $u \in C^2(\bar{\Omega})$, one derives from the Divergence Theorem (letting $w = Du$) that

$$\int_\Omega \Delta u \ dx = \int_{\partial\Omega} Du \bullet \nu \ dS = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \ dS. \tag{3.7}$$

Now, for u, $v \in C^2(\bar{\Omega})$, by choosing $w = uDv$ or $w = vDu$ respectively in the Divergence Theorem, we have

$$\int_\Omega u\Delta v\ dx = \int_{\partial\Omega} u\frac{\partial v}{\partial \nu}\ dS - \int_\Omega Du \bullet Dv\ dx \tag{3.8}$$

$$\int_\Omega v\Delta u\ dx = \int_{\partial\Omega} v\frac{\partial u}{\partial \nu}\ dS - \int_\Omega Dv \bullet Du\ dx. \tag{3.9}$$

We subtract the above two equations to get

$$\int_\Omega (u\Delta v - v\Delta u)\ dx = \int_{\partial\Omega} (u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu})\ dS. \tag{3.10}$$

Traditionally, (3.8) is called *the first Green's formula*, while (3.10) is called *the second Green's formula.*

3.1.2 Poisson Equation in $\mathbb{R}^n$

From the construction, we know that the fundamental solution $\Phi(x)$ of Laplace equation is harmonic for $x \neq 0$, so is $\Phi(x-y)$ for $x \neq y$. With this in mind, we will prove the following

Theorem 3.1.2 *If $f \in C_0^2(\mathbb{R}^n)$, then*

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)\ dy \in C(\mathbb{R}^n) \tag{3.11}$$

is a solution of the Poisson equation (3.2).

Proof. First of all, we have

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)\ dy = \int_{\mathbb{R}^n} \Phi(y)f(x-y)\ dy,$$

hence

$$\frac{u(x+he_i)-u(x)}{h} = \int_{\mathbb{R}^n} \Phi(y)\left[\frac{f(x+he_i-y)-f(x-y)}{h}\right]\ dy,$$

where $h \neq 0$ and e_i the unit vector in the direction of x_i-axis. Note that

$$\lim_{h\to 0}\frac{f(x+he_i-y)-f(x-y)}{h} = \frac{\partial f}{\partial x_i}(x-y)$$

uniformly on $\mathbb{R}^n$, and thus

$$\frac{\partial u}{\partial x_i}(x) = \int_{\mathbb{R}^n} \Phi(y)\frac{\partial f}{\partial x_i}(x-y)\ dy,\ \ (i = 1, \cdots, n). \tag{3.12}$$

Similarly,

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x) = \int_{\mathbb{R}^n} \Phi(y)\frac{\partial^2 f}{\partial x_i \partial x_j}(x-y)\ dy,\ \ (i, j = 1, \cdots, n). \tag{3.13}$$

We note that the integral above is continuous in the variable x, and thus $u \in C^2(\mathbb{R}^n)$.

As $\Phi(x)$ is singular at $x = 0$, we have to pay substantial attention near the singularity. Fix $\varepsilon > 0$, let $B(0, \varepsilon)$ be the ball centered at 0 with radius ε. We have

$$\begin{aligned}\Delta u(x) &= \int_{B(0,\varepsilon)} \Phi(y)\Delta_x f(x-y)\, dy + \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi(y)\Delta_x f(x-y)\, dy \\ &=: I_\varepsilon + J_\varepsilon.\end{aligned} \tag{3.14}$$

Now, it is clear that

$$I_\varepsilon \le C\|D^2 f\|_{L^\infty(\mathbb{R}^n)} \int_{B(0,\varepsilon)} |\Phi(y)|\, dy \le \begin{cases} C\varepsilon^2 |\log \varepsilon| & (n=2) \\ C\varepsilon^2 & (n=3). \end{cases} \tag{3.15}$$

On the other hand, we can estimate J_ε as following

$$\begin{aligned} J_\varepsilon &= \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi(y)\Delta_x f(x-y)\, dy \\ &= \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi(y)\Delta_y f(x-y)\, dy \\ &= -\int_{\mathbb{R}^n \setminus B(0,\varepsilon)} D_y\Phi(y) \bullet D_y f(x-y)\, dy + \int_{\partial B(0,\varepsilon)} \Phi(y)\frac{\partial f}{\partial v}(x-y)\, dS(y) \\ &=: K_\varepsilon + L_\varepsilon, \end{aligned} \tag{3.16}$$

where ν is the inward unit normal along $\partial B(0, \varepsilon)$. We easily check

$$|L_\varepsilon| \le C\|Df\|_{L^\infty(\mathbb{R}^n)} \int_{\partial B(0,\varepsilon)} |\Phi(y)|\, dy \le \begin{cases} C\varepsilon |\log \varepsilon| & (n=2) \\ C\varepsilon & (n=3). \end{cases} \tag{3.17}$$

It remains to compute K_ε. In fact, we have

$$\begin{aligned} K_\varepsilon &= -\int_{\mathbb{R}^n \setminus B(0,\varepsilon)} D_y\Phi(y) \bullet D_y f(x-y)\, dy \\ &= \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Delta_y\Phi(y) f(x-y)\, dy - \int_{\partial B(0,\varepsilon)} \frac{\partial \Phi}{\partial \nu}(y) f(x-y)\, dS(y) \\ &= -\int_{\partial B(0,\varepsilon)} \frac{\partial \Phi}{\partial \nu}(y) f(x-y)\, dS(y). \end{aligned}$$

We now note that

$$D\Phi(y) = \frac{-1}{n\alpha(n)} \frac{y}{|y|^n}, y \neq 0,$$

and

$$\nu = \frac{-y}{|y|} = -\frac{y}{\varepsilon}, \ \textit{on } \partial B(0, \varepsilon),$$

So,

$$\frac{\partial \Phi}{\partial \nu}(y) = \frac{1}{n\alpha(n)\varepsilon^{n-1}}, \;\; on\ \partial B(0,\varepsilon).$$

We also note that $n\alpha(n)\varepsilon^{n-1}$ is the surface area of $\partial B(0,\varepsilon)$, we have

$$\begin{aligned} K_\varepsilon &= -\frac{1}{n\alpha(n)\varepsilon^{n-1}} \int_{\partial B(0,\varepsilon)} f(x-y)\ dS(y) \\ &\to -f(x),\ \ as\ \varepsilon \to 0. \end{aligned} \tag{3.18}$$

We therefore conclude from (3.14)–(3.18) (letting $\varepsilon \to 0$) that

$$-\Delta u = f.$$

Remark 3.1.3 *Sometimes, we write*

$$-\Delta \Phi = \delta_0,\ \ in\ \mathbb{R}^n,$$

where δ_0 is the Dirac measure on $\mathbb{R}^n$ giving unit mass to the point 0. Formally, one can computes:

$$\begin{aligned} -\Delta u(x) &= \int_{\mathbb{R}^n} -\Delta_x \Phi(x-y) f(y)\ dy \\ &= \int_{\mathbb{R}^n} \delta_x f(y)\ dy = f(x),\ \ (x \in \mathbb{R}^n). \end{aligned}$$

3.1.3 Fundamental integral formulas

As we often solve the elliptic equations in bounded domains, we now carry out some useful integral formulas using the fundamental solution $\Phi(x)$.

Assume $u \in C^2(\Omega)$. For any $y \in \Omega$, we choose $\rho > 0$ suitably small so that the ball $B_\rho(y)$ centered at y with radius ρ is inside Ω. On the region $\Omega \setminus \bar{B}_\rho(y)$, we substitute $v(x)$ with $\Phi(x-y)$ in the *second Green's formula*,

$$\int_{\Omega - B_\rho(y)} \Phi(x-y)\Delta_x u\ dx = \int_{\partial\Omega} (\Phi\frac{\partial u}{\partial \nu} - u\frac{\partial \Phi}{\partial \nu})\ dS_x + \int_{\partial B_\rho(y)} (\Phi\frac{\partial u}{\partial \nu} - u\frac{\partial \Phi}{\partial \nu})\ dS_x. \tag{3.19}$$

Similar to the computations carried out in (3.17)–(3.18), we have

$$\begin{aligned} |\int_{\partial B_\rho(y)} \Phi\frac{\partial u}{\partial \nu}\ dS_x| &= |\Phi(\rho)\int_{\partial B_\rho(y)} \frac{\partial u}{\partial \nu}\ dS_x| \\ &= |-\Phi(\rho)\int_{B_\rho(y)} \Delta u\ dx| \\ &\le \begin{cases} C\rho^2|\log\rho| & (n=2) \\ C\rho^2 & (n=3). \end{cases} \\ &\to 0,\ \ as\ \rho \to 0; \end{aligned}$$

and

$$\begin{aligned}\int_{\partial B_\rho(y)} u\frac{\partial\Phi}{\partial\nu}\ dS_x &= -\Phi'(\rho)\int_{\partial B_\rho(y)} u\ dS_x\\ &= \frac{1}{n\alpha(n)\rho^{n-1}}\int_{\partial B_\rho(y)} u\ dS_x\\ &\to u(y),\ \ as\ \rho\to 0.\end{aligned}$$

Therefor, we obtain from (3.19) by letting $\rho \to 0$ that

$$u(y) = -\int_\Omega \Phi(x-y)\Delta_x u\ dx - \int_{\partial\Omega}(u\frac{\partial\Phi(x-y)}{\partial\nu} - \Phi(x-y)\frac{\partial u}{\partial\nu})\ dS_x,\ \forall y\in\Omega. \quad (3.20)$$

This formula is called *Green's representation* of $u(y)$.

In particular, if u has compact support on Ω, then it holds

$$u(y) = -\int_\Omega \Phi(x-y)\Delta_x u\ dx,\ \forall y\in\Omega,\ u\in C_0^2(\Omega). \quad (3.21)$$

If u is harmonic in Ω, we thus obtain the *fundamental integral formula* of harmonic functions

$$u(y) = -\int_{\partial\Omega}(u\frac{\partial\Phi(x-y)}{\partial\nu} - \Phi(x-y)\frac{\partial u}{\partial\nu})\ dS_x,\ \forall y\in\Omega. \quad (3.22)$$

We remark that, actually, for smooth harmonic function, (3.7) implies

$$\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\ dS = 0. \quad (3.23)$$

3.2 Properties of harmonic functions

In this section, we discuss some basic properties of harmonic functions. We now consider an open bounded set $\Omega\subset\mathbb{R}^n$ and suppose u is harmonic in Ω. Various interesting properties will be presented in orders.

3.2.1 Mean-value formulas

The *mean-value formulas*, which declare that $u(x)$ equals both the average of u over the sphere $\partial B(x,r)$ and the average of u over the whole ball $B(x,r)$, as long as $B(x,r)\subset\Omega$. It will play key roles in many important occasions.

To begin, we introduce the notion of *mean value* of $u(x)$ over a domain Ω:

$$(u)_\Omega = \frac{1}{|\Omega|}\int_\Omega u(x)\ dx. \quad (3.24)$$

Theorem 3.2.1 *If $u\in C^2(\Omega)$ is harmonic, then*

$$u(x) = (u)_{\partial B(x,r)} = (u)_{B(x,r)}, \quad (3.25)$$

for each ball $B(x,r)\subset\Omega$.

Proof. By (3.22), using (3.23), one has

$$\begin{aligned} u(x) &= -\int_{\partial B(x,r)} (u\frac{\partial \Phi(x-y)}{\partial \nu} - \Phi(x-y)\frac{\partial u}{\partial \nu})\ dS_y \\ &= -\Phi'(r)\int_{\partial B(x,r)} u(y)\ dS_y - \Phi(r)\int_{\partial B(x,r)} \frac{\partial u}{\partial \nu}\ dS_y \\ &= -\Phi'(r)\int_{\partial B(x,r)} u(y)\ dS_y \\ &= (u)_{\partial B(x,r)}. \end{aligned}$$

Now, for the mean-value on the ball, we have

$$\begin{aligned} \int_{B(x,r)} u(y)\ dy &= \int_0^r (\int_{\partial B(x,s)} u(y)\ dS_y)\ ds \\ &= u(x)\int_0^r \int_{\partial B(x,s)} dS_y\ ds = u(x)\cdot(\mathit{volume\ of}\ B(x,r)). \end{aligned}$$

This completes the proof.

Remark 3.2.2 If instead of harmonic, $u \in C^2(\Omega)$ is sub-harmornic, then one has

$$u(x) \le (u)_{\partial B(x,r)},\ u(x) \le (u)_{B(x,r)}. \tag{3.26}$$

If $u \in C^2(\Omega)$ is super-harmornic, then

$$u(x) \ge (u)_{\partial B(x,r)},\ u(x) \ge (u)_{B(x,r)}. \tag{3.27}$$

The converse of the *mean-value formulas* is also a true statement.

Theorem 3.2.3 *If $u \in C^2(\Omega)$ satisfies*

$$u(x) = (u)_{\partial B(x,r)}$$

for each ball $B(x,r) \subset \Omega$, then u is harmonic.

Proof. Set

$$\phi(r) = (u)_{\partial B(x,r)} = (u(x+rz))_{\partial B(0,1)}.$$

Then

$$\begin{aligned} \phi'(r) &= (Du(x+rz)\bullet z)_{\partial B(0,1)} \\ &= \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} Du(y)\bullet\frac{y-x}{r}\ dS_y \\ &= \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} \frac{\partial u}{\partial \nu}\ dS_y \\ &= \frac{r}{n}(\Delta u)_{B(x,r)} \end{aligned}$$

If $\Delta u \neq 0$, there exists some ball $B(x, r_1) \subset \Omega$ such that $\Delta u > 0$ in $B(x, r)$ (the other case is similar). But

$$0 = \phi'(r_1) = \frac{r_1}{n}(\Delta u)_{B(x,r_1)} > 0,$$

a contradiction.

3.2.2 Maximum principle and uniqueness

Assume that Ω is an open and bounded set in $\mathbb{R}^n$. We first present

Theorem 3.2.4 *(Strong maximum principle). Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is harmonic in Ω, then*

- *(i)* $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$
- *(ii) If Ω is connected and there exists a point $x_0 \in \Omega$ such that*

  $$u(x_0) = \max_{\bar{\Omega}} u,$$

  then u is constant in Ω.

Remark 3.2.5 The first assertion in this theorem is the *maximum principle* and the second is the *strong maximum principle.* If one replaces u by $-u$, the similar assertions are true when "max" is replaced by "min".

Proof. Suppose there exists a point $x_0 \in \Omega$ with

$$u(x_0) = M = \max_{\bar{\Omega}} u.$$

Then for some $r > 0$, the ball $B(x_0, r) \subset \Omega$. By the mean-value formula on this ball, we have

$$M = u(x_0) = (u)_{B(x_0,r)} \leq M.$$

Here, the equality holds only if $u \equiv M$ in $B(x_0, r)$. Therefore, $u(x) = M$ for any $x \in B(x, r)$. So, the set $\{x \in \Omega | u(x) = M\}$ is both open and relatively closed in Ω, and thus equals Ω if Ω is connected. This proves assertion (ii). The first one follows immediately.

The strong maximum principle has many applications. A direct application is as follows: if Ω is connected and $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is a solution of

$$\begin{cases} \Delta u = 0, \ \textit{in } \Omega, \\ u = g, \ \textit{on } \partial\Omega, \end{cases}$$

where $g \geq 0$, then u is positive everywhere in Ω if g is positive somewhere on $\partial\Omega$.

An important application of the maximum principle is the uniqueness of solutions to certain boundary value problems for Poisson equation.

Theorem 3.2.6 *(Uniqueness). Let $g \in C(\partial\Omega)$, $f \in C(\Omega)$. Then there exists at most one solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ of the boundary value problem*

$$\begin{cases} -\Delta u = f, \ in \ \Omega, \\ u = g, \ on \ \partial\Omega. \end{cases} \tag{3.28}$$

Proof. If u_1 and u_2 are two solutions, one applies the maximum principle to the harmonic functions $w_\pm = \pm(u_1 - u_2)$ which satisfies the Laplace equation with zero boundary condition, therefore, $w_\pm \equiv 0$.

3.2.3 Harnack's inequality

In this subsection, we will exploit an amazing averaging effect of Harmonic functions. Recall that we denote by $V \subset\subset \Omega$ if $\bar{V} \subset \Omega$ is compact. The following theorem assert that the value of a non-negative harmonic function on Ω are all comparable within a compact subset relative to Ω: it cannot be very large (or very small) at any point in this subset unless it is very large (or very small) everywhere. The idea is that since the compact set has positive distance from $\partial\Omega$, there is *room for the averaging effect of Laplace's equation to occur.*

Theorem 3.2.7 *(Harnack's inequality). For each connecterd open set $V \subset\subset \Omega$, there exists a positive constant C, depending only on V, such that*

$$\sup_V u \le C \inf_V u$$

for all nonnegative harmonic functions u in Ω.

Remark 3.2.8 The Harnack's inequality in particular implies that, for any x and $y \in V$, it holds that

$$\frac{1}{C} u(y) \le u(x) \le C \ u(y).$$

Proof. Let $r = \frac{1}{4} \, dist(V, \partial\Omega)$. Fixing any $x \in V$, for any $y \in V$ such that $|x - y| \le r$, we have

$$\begin{aligned} u(x) &= (u)_{B(x,2r)} \\ &\ge \frac{1}{\alpha(n)2^n r^n} \int_{B(y,r)} u \ dz \\ \frac{1}{2^n}(u)_{B(y,r)} &= \frac{1}{2^n} u(y). \end{aligned}$$

Therefore, we actually proved that

$$2^{-n} u(y) \le u(x) \le 2^n u(x), \forall x, y \in V, \ |x - y| \le r.$$

Since V is connected and $\bar{V}$ is compact, we can cover $\bar{V}$ by a chain of finitely many balls $\{B_i\}_{i=1}^N$, each of which has radius r and $B_i \cap B_{i-1} \neq \emptyset$, for $i = 2, \cdots, N$. Then,

$$2^{-nN} u(y) \le u(x) \le 2^{nN} u(y)$$

for all $x, y \in V$.

3.2.4 Regularity

Now we prove that if $u \in C^2(\Omega)$ is harmonic, then necessarily $u \in C^\infty(\Omega)$. Thus *harmonic functions are infinitely differentiable*. This sort of assertion is called a *regularity* statements. It is interesting to see that the algebraic structure of Laplace equation leads to that all the partial derivatives of u exist, even those which do not appear in the PDE.

Theorem 3.2.9 *(Smoothness). If $u \in C(\Omega)$ satisfies the mean-value property (3.25) for each ball $B(x,r) \subset \Omega$, then $u \in C^\infty(\Omega)$.*

Before we proceed to prove this theorem, we first introduce an important tool. Define $\eta \in C^\infty(\mathbb{R}^n)$ by

$$\eta(x) = \begin{cases} C \exp\{\frac{1}{|x|^2-1}\} & if\ |x| < 1, \\ 0 & if\ |x| > 1, \end{cases} \tag{3.29}$$

where $C > 0$ is chosen so that

$$\int_{\mathbb{R}^n} \eta(x)\ dx = 1.$$

Now, for each $\varepsilon > 0$, set

$$\eta_\varepsilon = \varepsilon^{-n} \eta(\frac{x}{\varepsilon}).$$

Such η is called the *standard mollifier*, and $\eta_\varepsilon \in C^\infty(\mathbb{R}^n)$ statisfies

$$\int_{\mathbb{R}^n} \eta_\varepsilon(x)\ dx = 1, \quad supp(\eta_\varepsilon) \subset B(0, \varepsilon). \tag{3.30}$$

The following lemma states some of the properties of mollifiers.

Lemma 3.2.10 *If $f : \Omega \to \mathbb{R}$ is locally integrable, define*

$$f^\varepsilon = \eta_\varepsilon \star f = \int_\Omega \eta_\varepsilon(x-y) f(y)\ dy,$$

for $x \in \Omega_\varepsilon = \{x \in \Omega |\ dist(x, \partial\Omega) > \varepsilon\}$. Then, it holds that

- *(i) $f^\varepsilon \in C^\infty(\Omega_\varepsilon)$.*
- *(ii) $f^\varepsilon \to f$, a.e. as $\varepsilon \to 0$.*
- *(iii) If $f \in C(\Omega)$, then $f^\varepsilon \to f$ uniformly on compact subsets of Ω.*

We now give a proof to the regularity theorem using the mean-value formulas.
Proof. Let η be a standard mollifier, we will prove that $u = u^\varepsilon$ on Ω_ε. In fact, if $x \in \Omega_\varepsilon$, then

$$\begin{aligned}
u^\varepsilon(x) &= \int_\Omega \eta_\varepsilon(x-y)u(y)\ dy \\
&= \varepsilon^{-n}\int_B (x,\varepsilon)\eta(\frac{|x-y|}{\varepsilon})u(y)\ dy \\
&= \varepsilon^{-n}\int_0^\varepsilon \eta(\frac{r}{\varepsilon})(\int_{\partial B(x,r)} u(y)\ dS)\ dr \\
&= \varepsilon^{-n}u(x)\int_0^\varepsilon \eta(\frac{r}{\varepsilon})n\alpha(n)r^{n-1}\ dr \\
&= u(x)\int_{B(0,\varepsilon)} \eta_\varepsilon\ dy = u(x).
\end{aligned}$$

Therefore, $u \in C^\infty(\Omega_\varepsilon)$ for any $\varepsilon > 0$, and so $u \in C^\infty(\Omega)$.

Remark 3.2.11 One should be careful that u is not assumed to have any regularity at boundary, even continuity.

A further applications of mean-value formulas will lead to the estimates on derivatives, for which we omit the proof.

Theorem 3.2.12 *(Estimates on derivatives). Assume u is harmonic in Ω. Then it holds that*

$$|D^\alpha u(x_0)| \le \frac{C_k}{r^{n+k}}\|u\|_{L^1(B(x_0,r))} \tag{3.31}$$

for each ball $B(x_0, r) \subset \Omega$ and each multi-index α of order $|\alpha| = k$. Here, the C_k can be chosen as

$$C_0 = \frac{1}{\alpha(n)},\ C_k = \frac{(2^{n+1}nk)^k}{\alpha(n)},\ k = 1, \cdots . \tag{3.32}$$

The C^∞ regularity does not go to extreme of the harmonic functions. The following theorem states that *harmonic functions are actually analytic.*

Theorem 3.2.13 *If u is harmonic in Ω, then u is analytic in Ω.*

The next one confirms the analyticity of harmonic functions, which says that there are no nontrivial bounded harmonic functions on whole $\mathbb{R}^n$.

Theorem 3.2.14 *(Liouville's Theorem). If $u : \mathbb{R}^n \to \mathbb{R}$ is harmonic and bounded, then u is constant.*

Proof. Fix $x_0 \in \mathbb{R}^n$, $r > 0$, and apply Theorem 3.2.12 on $B(x_0, r)$,

$$\begin{aligned} |Du(x_0)| &\leq \frac{C_1}{r^{n+1}} \|u\|_{L^1(B(x_0,r))} \\ &\leq \frac{C_1 \alpha(n)}{r} \|u\|_{L^\infty(\mathbb{R}^n)} \to 0, \ \textit{as } r \to \infty. \end{aligned}$$

Therefore, $Du \equiv 0$, and so u is constant.

As a direct application of Liouville's Theorem, we have

Theorem 3.2.15 *(Representation formula). Let $f \in C_0^2(\mathbb{R}^n)$, $n \geq 3$. Then any bounded solution of*

$$-\Delta u = f \ \textit{in } \mathbb{R}^n$$

has the form

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) \ dy + C, \ \ (x \in \mathbb{R}^n)$$

for some constant C.

Proof. Since $\Phi(x) \to 0$ as $|x| \to \infty$ for $n \geq 3$,

$$\tilde{u}(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) \ dy$$

is a bounded solution of

$$-\Delta u = f \ \textit{in } \mathbb{R}^n.$$

If u is another bouned solution, $w = u - \tilde{u}$ is bounded and harmonic, and thus is a constant.

Remark 3.2.16 When $n = 2$, $\Phi(x)$ is not bounded as $|x| \to \infty$, and so it is possible that

$$\int_{\mathbb{R}^n} \Phi(x-y) f(y) \ dy$$

is not bounded as $|x| \to \infty$. Therefore, the representation formula is not true in general for $n = 2$.

3.3 Green's function

In last section, we obtained representation formula for problems on $\mathbb{R}^n$. We now fix Ω to be an bounded open domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. We will try to find a general representation formula for the solutions of the following boundary value problem

$$\begin{cases} -\Delta u = f \ \textit{in } \Omega, \\ u = g \ \textit{on } \partial\Omega. \end{cases} \tag{3.33}$$

We intend to solve this problem through the Green's representation formula (3.20). We recall (3.20) below

$$u(x) = -\int_{\Omega} \Phi(x-y)\Delta_y u(y)\, dy - \int_{\partial\Omega} (u(y)\frac{\partial \Phi(x-y)}{\partial \nu} - \Phi(x-y)\frac{\partial u(y)}{\partial \nu})\, dS_y,\ \forall x \in \Omega. \tag{3.34}$$

It is clear that (3.34) permits us to solve the problem if we know how to deal with the term of $\frac{\partial u}{\partial \nu}$ on the boundary. Unfortunately, it is unknown to us. The idea is to introduce a *correction* $h(x, y)$ for each fixed x such that it solves the following boundary value problem.

$$\begin{cases} -\Delta_y h = 0 \ in\ \Omega, \\ h = \Phi(x-y) \ on\ \partial\Omega. \end{cases} \tag{3.35}$$

By the Green's formula, we have

$$\begin{aligned} -\int_{\Omega} h(x,y)\Delta_y u(y)\ dy &= \int_{\partial\Omega} (u(y)\frac{\partial h(x,y)}{\partial \nu} - h(x,y)\frac{\partial u}{\partial \nu})\ dS_y \\ &= \int_{\partial\Omega} (u(y)\frac{\partial h(x,y)}{\partial \nu} - \Phi(x-y)\frac{\partial u}{\partial \nu})\ dS_y. \end{aligned} \tag{3.36}$$

We therefore arrived at the following definition.

Definition 3.3.1 *Green's function for the region Ω is*

$$G(x,y) = \Phi(x-y) - h(x,y),\ (x,y \in \Omega,\ x \neq y).$$

With this notion, we add (3.36) to (3.34) to find

$$u(x) = -\int_{\Omega} G(x,y)\Delta_y u(y)\ dy - \int_{\partial\Omega} u(y)\frac{\partial G(x,y)}{\partial \nu}\ dS_y,\ \forall x \in \Omega. \tag{3.37}$$

where

$$\frac{\partial G}{\partial \nu}(x,y) = \nabla_y G(x,y) \cdot \nu(y),$$

is the outer normal derivative of G with respect to variable y. The term $\frac{\partial u}{\partial \nu}$ does not appear in this formula (3.37). This means the correction $h(x, y)$ is in its proper form.

Now, suppose $u(x) \in C^2(\bar{\Omega})$ solves the boundary value problem (3.33) for some continuous functions f and g, using (3.37), we arrive at

Theorem 3.3.2 *(Representation formula with Green's function). If $u(x) \in C^2(\bar{\Omega})$ solves problem (3.33), then*

$$u(x) = \int_{\Omega} G(x,y)f(y)\ dy - \int_{\partial\Omega} g(y)\frac{\partial G(x,y)}{\partial \nu}\ dS_y,\ \forall x \in \Omega. \tag{3.38}$$

Remark 3.3.3 Formally, (3.38) gives a nice formula to the solutions of the Poisson equation with Dirichlet boundary condition if we know the Green's function on the domain Ω, such a method is called *Green's method.* However, for general domain, it is a difficult task to construct the Green's function, for which requires to solve the problem (3.35). However, Green's method is still significant in the following reasons:

- (i) For Poisson equation on a fixed domain, once we obtained the Green's function, the existence of solutions to the problem (3.33) is given by the formula (3.38) for any continuous f and g.
- (ii) In the case when it is hard to find the solutions to (3.33), one can still use the formula (3.38) to discuss the certain behavior of the solutions.
- (iii) For some domains with simple geometry, explicit calculation of G is possible. The Dirichlet problem of Poisson equation on such domains are often important in the applications.
- (iv) (3.38) transfers the (3.33) into a integral equation, which is convenient in certain occasions even when the equation is semi-linear. The machinery of functional analysis is thus applied to obtain some interesting results.

Before going to specific examples, we discuss certain important properties of Green's function.

First of all, it is clear that in Ω, when $x \neq y$, $G(x, y)$ is harmonic on x and on y everywhere. Furthermore, $G(x, y) \to \infty$ as $x \to y$ at the order of $|x - y|^{n-2}$ if $n > 2$ and at the order of $\log |x - y|$ if $n = 2$.

Secondly, substituting $u(x) = 1$ into (3.37) we have

$$\int_{\partial\Omega} \frac{\partial G}{\partial \nu}\, dS = -1. \tag{3.39}$$

Finally, we show that $G(x, y)$ is symmetric in x and y.

Theorem 3.3.4 *(Symmetry of Green's function). For all x, $y \in \Omega$, $x \neq y$, it holds that*

$$G(y, x) = G(x, y).$$

Proof. Formally, we prove the theorem as following. For $x \neq y \in \Omega$, $\Phi(|x - y|)$ is smooth on $\partial\Omega$, by definition, we know that

$$G(y, x) = \Phi(x - y) - h(x, y),\ G(y, x) = \Phi(y - x) - h(y, x).$$

We note that $\Phi(x-y) = \Phi(y-x) = \Phi(|x-y|)$, and therefore, both $h(x, y)$ and $h(y, x)$ are the harmonic solutions of the same problem (3.35). By uniqueness we know $h(x, y) = h(y, x)$ and therefore

$$G(y, x) = G(x, y),\ \forall x \neq y \in \Omega.$$

3.3.1 Green's function on a ball

We will construct Green's function for the unit ball $B(0,1)$ using some reflection through the sphere $\partial B(0,1)$.

Definition 3.3.5 *If $x \in \mathbb{R}^n \setminus \{0\}$, the point*

$$\bar{x} = \frac{x}{|x|^2}$$

is called the point dual to x with respect to $\partial B(0,1)$. The mapping $x \to \bar{x}$ is inversion through the unit sphere $\partial B(0,1)$.

We will use this inversion to construct Green's function for the unit ball $\Omega = B(0,1)$. Fix $x \in B(0,1)$. We need to find a correction $h(x,y)$ solving

$$\begin{cases} -\Delta_y h = 0 \text{ in } B(0,1), \\ h = \Phi(x-y) \text{ on } \partial B(0,1), \end{cases} \tag{3.40}$$

then the Green's function reads

$$G(x,y) = \Phi(x-y) - h(x,y).$$

We try to *invert the singularity* of $\Phi(x-y)$ from $x \in B(0,1)$ to $\bar{x} \notin B(0,1)$. Assume now that $n \geq 3$. The mapping

$$y \to \Phi(y - \bar{x})$$

is harmonic for $y \neq \bar{x}$. Thus,

$$y \to |x|^{2-n}\Phi(y-\bar{x})$$

is harmonic for $y \neq \bar{x}$, and so

$$h(x,y) = \Phi(|x|(y-\bar{x})) \tag{3.41}$$

is harmonic in $B(0,1)$. Furthermore, if $y \in \partial B(0,1)$ and $x \neq 0$,

$$\begin{aligned} |x|^2|y-\bar{x}|^2 &= |x|^2\left(|y^2 - \frac{2y\cdot x}{|x|^2} + \frac{1}{|x|^2}\right) \\ &= |x|^2 - 2y\cdot x + 1 = |x-y|^2. \end{aligned}$$

Therefore, $(|x||y-\bar{x}|)^{2-n} = |x-y|^{2-n}$ and so

$$h(x,y) = \Phi(y-x), \ \forall y \in \partial B(0,1). \tag{3.42}$$

This verifies that $h(x,y)$ is the one we were looking for.

Definition 3.3.6 *Green's function for the unit ball is*

$$G(x,y) = \Phi(y-x) - \Phi(|x|(y-\bar{x})),\ x,y \in B(0,1),\ x \neq y. \tag{3.43}$$

We now solve the following boundary value problem

$$\begin{cases} \Delta u &= 0,\ in\ \ B(0,1) \\ u &= g,\ on\ \ \partial B(0,1). \end{cases} \tag{3.44}$$

By (3.38), we need to calculate $\dfrac{\partial G}{\partial \nu}$ on the unit sphere. According to formula (3.43), for $y \in \partial B(0,1)$ one has

$$\frac{\partial G}{\partial y_i} = \frac{\partial \Phi(y-x)}{\partial y_i} - \frac{\partial \Phi(|x|(y-\bar{x}))}{\partial y_i},$$

where

$$\frac{\partial \Phi(y-x)}{\partial y_i} = \frac{-1}{n\alpha(n)} \frac{y_i - x_i}{|x-y|^n},$$

and

$$\frac{\partial \Phi(|x|(y-\bar{x}))}{\partial y_i} = \frac{-1}{n\alpha(n)} \frac{y_i|x|^2 - x_i}{(|x||y-\bar{x}|)^n} = \frac{-1}{n\alpha(n)} \frac{y_i|x|^2 - x_i}{|x-y|^n},\ \forall y \in \partial B(0,1).$$

Therefore,

$$\begin{aligned} \frac{\partial G}{\partial \nu} &= \sum_{i=1}^{n} y_i \frac{\partial G}{\partial y_i}(x,y) \\ &= \frac{-1}{n\alpha(n)} \frac{1}{|x-y|^n} \sum_{i=1}^{n} y_i((y_i - x_i) - y_i|x|^2 + x_i) \\ &= \frac{-1}{n\alpha(n)} \frac{1-|x|^2}{|x-y|^n},\ \forall y \in \partial B(0,1). \end{aligned}$$

Thus, we use (3.38) to yield the representation formula

$$u(x) = \frac{1-|x|^2}{n\alpha(n)} \int_{\partial B(0,1)} \frac{g(y)}{|x-y|^n}\, dS_y. \tag{3.45}$$

If now instead of (3.43), for $r > 0$, we want to solve the following boundary-value problem

$$\begin{cases} \Delta u &= 0,\ in\ \ B(0,r) \\ u &= g,\ on\ \ \partial B(0,r). \end{cases} \tag{3.46}$$

Then, $\bar{u}(x) = u(rx)$ solves (3.43) with $\bar{g}(x) = g(rx)$ replacing g in (3.43). After a direct change of variables, we obtain the *Poisson's formula*

$$u(x) = \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\, dS_y,\ \forall x \in B(0,1). \tag{3.47}$$

The function

$$K(x,y) = \frac{r^2 - |x|^2}{n\alpha(n)r} \frac{1}{|x-y|^n}, \; x \in B(0,r), \; y \in \partial B(0,r), \tag{3.48}$$

is called *Poisson's Kernel* for the ball $B(0,r)$.

(3.47) was established under the assumption that (3.46) has a smooth solution. We now prove that in fact (3.47) gives a solution.

Theorem 3.3.7 *(Poisson's formula for a ball). Assume $g \in C(\partial B(0,r))$ and u is defined by (3.47). Then u is harmonic in $B(0,r)$ and for each point $x_0 \in \partial B(0,r)$,*

$$\lim_{x \to x_0} u(x) = g(x_0), \; x \in B(0,r).$$

Proof. It is clear that $K(x,y) \geq 0$ is harmonic when $x \neq y$. Therefore, for $x \in B(0,r)$ and $y \in \partial B(0,r)$, we have

$$\Delta u(x) = \int_{\partial B(0,r)} \Delta_x K(x,y) g(y) \; dS_y = 0.$$

To verify the boundary condition, we first remark that

$$\int_{\partial B(0,r)} K(x,y) \; dS_y = 1. \tag{3.49}$$

Indeed, for each fixed $y \in \partial B(0,r)$, we denote $x = \sigma z$ where $0 \leq \sigma < 1$ and $z \in \partial B(0,r)$. By the mean-value formula for harmonic function, we have

$$1 = K(0,y) n\alpha(n) r^{n-1} = \int_{\partial B(0,r)} K(\sigma z, y) \; dS_z.$$

Now,

$$\begin{aligned} 1 &= \int_{\partial B(0,r)} K(\sigma z, y) \; dS_z \\ &= \int_{\partial B(0,r)} K(\sigma y, z) \; dS_z \\ &= \int_{\partial B(0,r)} K(x, z) \; dS_z. \end{aligned}$$

Now, fix $x_0 \in \partial B(0,r)$, $\varepsilon > 0$. Choose $\delta > 0$ so small that

$$|g(y) - g(x_0)| < \varepsilon, \text{ if } \; |y - x_0| < \delta, \; y \in \partial B(0,r). \tag{3.50}$$

Then, if $|x - x_0| < \frac{\delta}{2}$, $x \in B(0,r)$, setting

$$V_\delta = \partial B(0,r) \cap B(x_0, \delta),$$

we compute

$$
\begin{aligned}
|u(x) - g(x_0)| &= \left| \int_{\partial B(0,r)} K(x,y)[g(y) - g(x_0)]\ dS_y \right| \\
&\le \int_{V_\delta} K(x,y)|g(y) - g(x_0)|\ dS_y \\
&\quad + \int_{\partial B(0,r)\setminus V_\delta} K(x,y)|g(y) - g(x_0)|\ dS_y \\
&\equiv I + J.
\end{aligned} \tag{3.51}
$$

Now, (3.49)–(3.50) implies that

$$
I \le \varepsilon \int_{\partial B(0,r)} K(x,y)\ dS_y = \varepsilon.
$$

For J, we note that if $|x - x_0| < \frac{\delta}{2}$ and $|y - x_0| \ge \delta$, then

$$
|y - x| \ge \frac{1}{2}|y - x_0|.
$$

Thus,

$$
\begin{aligned}
J &\le 2\|g\|_{L^\infty} \int_{\partial B(0,r)\setminus V_\delta} K(x,y)\ dS_y \\
&\le \frac{2^{n+1}\|g\|_{L^\infty}(r^2 - |x|^2)}{n\alpha(n)r} \int_{\partial B(0,r)\setminus V_\delta} |y - x_0|^{-n}\ dS_y \\
&\to 0, \quad as \quad x \to x_0.
\end{aligned}
$$

Therefore, we could choose $\delta > \delta_\varepsilon > 0$ so small such that

$$
|u(x) - g(x_0)| < 2\varepsilon,\ if \quad |x - x_0| < \delta_\varepsilon.
$$

This proves that

$$
\lim_{x \to x_0} u(x) = g(x_0),\ x \in B(0,r).
$$

3.3.2 Green's function on half space

Again, we fix $n \ge 3$. Let us consider the half space

$$
\mathbb{R}^n_+ = \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n | x_n > 0\}.
$$

Although this region is unbounded, and so the calculations for Theorem 3.3.2 is not valid. We will try to build Green's function using the ideas developed so far. Later, we will check directly that the derived representation formula gives the solution. We will also use the reflection idea about the boundary of the domain.

Definition 3.3.8 *If* $x = (x_1, \cdots, x_{n-1}, x_n) \in \mathbb{R}^n_+$*, its reflection in the plane* $\partial\mathbb{R}^n_+$ *is the point*

$$\bar{x} = ((x_1, \cdots, x_{n-1}, -x_n).$$

We set

$$h(x,y) = \Phi(y-\bar{x}) = \Phi(y_1 - x_1, \cdots, y_{n-1} - x_{n-1}, y_n + x_n), \ (x, y \in \mathbb{R}^n_+).$$

we note that

$$h(x,y) = \Phi(y-x), \ if \ y \in \partial\mathbb{R}^n_+,$$

and hence

$$\begin{cases} \Delta h(x,y) = 0 & in \ \mathbb{R}^n_+ \\ h(x,y) = \Phi(y-x) & on \ \partial\mathbb{R}^n_+. \end{cases}$$

We thus has

Definition 3.3.9 *Green's function for the half-space* $\mathbb{R}^n_+$ *is*

$$G(x,y) = \Phi(y-x) - \Phi(y-\bar{x}), \ (x, y \in \mathbb{R}^n_+, \ x \neq y).$$

Clearly, if $y \in \partial\mathbb{R}^n_+$,

$$\frac{\partial G}{\partial \nu}(x,y) = -\frac{\partial G}{\partial y_n}(x,y) = \frac{2x_n}{n\alpha(n)} \frac{1}{|x-y|^n}.$$

Suppose u is a solution to the boundary-value problem

$$\begin{cases} \Delta u = 0 \ in \ \mathbb{R}^n_+ \\ u = g \ on \ \partial\mathbb{R}^n_+. \end{cases} \tag{3.52}$$

Then, formally, we expect from (3.38) that

$$u(x) = \int_{\partial\mathbb{R}^n_+} K(x,y) g(y) \, dy, \ (x \in \mathbb{R}^n_+) \tag{3.53}$$

to be a representation formula for the solution. Here the function

$$K(x,y) = \frac{2x_n}{n\alpha(n)} \frac{1}{|x-y|^n}, \quad (x, y \in \mathbb{R}^n_+, \ x \neq y), \tag{3.54}$$

is the *Poisson's kernel* for $\mathbb{R}^n_+$ and (3.53) is called the *Poisson's formula.*

Similar to the case for a ball, we could verify directly that

Theorem 3.3.10 *(Poisson's formula for half-space). Assume* $g \in C(\mathbb{R}^{n-1} \cap L^\infty(\mathbb{R}^{n-1})$*, and u is defined by (3.53). Then u is uniformly bounded harmonic function on* $\mathbb{R}^n_+$ *and for each* $x_0 \in \partial\mathbb{R}^n_+$*, it holds that*

$$\lim_{x \to x_0} u(x) = g(x_0), \ x \in \mathbb{R}^n_+.$$

3.4 Hopf's maximum principle

We have put a lot of efforts in the last section for the Dirichlet boundary value problem for Poisson equation. The Green's function method is particularly designed for this type of problems. The second boundary value problem, namely the Nuemann problem, is not well studied. We will establish the maximum principle for Nuemann problem in this section.

Theorem 3.4.1 *(Hopf's Lemma) Let $B(y,R) \subset \mathbb{R}^n$ $(n \geq 3)$, $x_0 \in \partial B(y,R)$, $u \in C^2(B(y,R)) \cap C^1(\bar{B}(y,R))$ is sub-harmonic on $B(y,R)$ such that*

$$u(x_0) > u(x), \ \forall x \in B(y,R),$$

then

$$\frac{\partial u}{\partial \nu}(x_0) > 0,$$

where ν is the outer unit normal of $\partial B(y,R)$ at x_0.

Proof. For $\rho \in (0,R)$, and a positive parameter $\alpha > 0$, we define

$$v(x) = e^{-\alpha r^2} - e^{-\alpha R^2}, \quad r = |x-y| > \rho.$$

Direct computation gives

$$\Delta v = e^{-\alpha r^2}(4\alpha^2 r^2 - 2na).$$

Therefore, if we choose α big enough, say $\alpha = \frac{n}{\rho}$, then $\Delta v \geq 0$ on the region $\mathcal{A} = B(y,R) \setminus \bar{B}(y,\rho)$. We note that $u(x) - u(x_0) < 0$ on $\partial B(y,\rho)$, there exists a $\varepsilon > 0$ such that

$$w(x) = u(x) - u(x_0) + \varepsilon v(x) \leq 0, \ x \in \partial B(y,\rho).$$

We note that $v(x) = 0$ on $\partial B(y,R)$ and thus

$$w(x) \leq 0, \ x \in \partial B(y,\rho).$$

We also note that $w(x)$ is sub-harmonic on $\mathcal{A}$, therefore the maximum principle for w on $\mathcal{A}$ implies that

$$w(x) \leq 0, \ \forall x \in \mathcal{A}.$$

But $w(x_0) = 0$ and thus

$$\frac{\partial u}{\partial \nu}(x_0) \geq 0,$$

that is

$$\frac{\partial u}{\partial \nu}(x_0) \geq -\varepsilon \frac{\partial v}{\partial \nu}(x_0) = -\varepsilon v'(R) > 0.$$

We now introduce a concept concerning the structure of the boundary.

Definition 3.4.2 *Let $x_0 \in \partial\Omega$, if there exists a ball $B \subset \Omega$ such that $\{x_0\} = \bar{B} \cap \bar{\Omega}$, we say Ω satisfies the inner ball condition at x_0. Meanwhile, we say $\mathbb{R}^n \setminus \bar{\Omega}$ satisfies the outer ball condition at x_0.*

With this notion, based on Hopf's lemma, we are able to prove the following

Theorem 3.4.3 *(Hopf's maximum principle) Assume $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ and $-\Delta u \leq 0$ ($-\Delta u \geq 0$), and $x_0 \in \partial\Omega$ such that*

$$u(x_0) > u(x) \quad (u(x_0) < u(x)), \ \forall x \in \Omega.$$

If Ω staisfies the inner ball condition at x_0, then

$$\frac{\partial u}{\partial \nu}(x_0) > 0 \quad (\frac{\partial u}{\partial \nu}(x_0) < 0).$$

We now apply Hopf's maximum principle to the Neumann problem. Consider

$$\begin{cases} -\Delta u = f, \ in \ \Omega \\ \frac{\partial u}{\partial \nu} = g, \ on \ \partial\Omega. \end{cases} \tag{3.55}$$

It is easy to see that the solution of the above Neumann problem, if exists, is not unique. For if u is one solution, then $u + C$ for any constant C is another solution. However, we could prove the following

Theorem 3.4.4 *If Ω satisfies the inner ball condition at each boundary point, then the solutions of Neumann problem (3.55) can only differ by a constant.*

Proof. Let u_1 and u_2 be two solutions to (3.55), then $w = u_1 - u_2$ is the solution of

$$\begin{cases} -\Delta w = f, \ in \ \Omega \\ \frac{\partial w}{\partial \nu} = 0, \ on \ \partial\Omega. \end{cases}$$

Now, if w is not constant, by the maximum principle for harmonic function w, we know w attains its maximum at some $x_0 \in \partial\Omega$. According to Hopf's maximum principle, we know

$$\frac{\partial w}{\partial \nu}(x_0) > 0$$

which contradicts the boundary condition. So w is a constant.

So far, we proved the uniqueness and stability for Dirichlet problem by maximum principle, showed the relative uniqueness for Neumann problem via Hopf's maximum principle. For the Dirichlet problem of Laplace equation on some special domains (such as a ball, and half space), we constructed Green's functions and thus gave the existence for continuous boundary data. However, the solvability of Dirichlet and Neumann problems on general domains are not clear yer. These, however, will require the concept of weak solutions, for which the functional analysis will play central role. This topic will be presented later.

3.5 Examples

In the first two examples we construct the Green's functions for the disk and upper half plane in $\mathbb{R}^2$.

Example 3.5.1 Find the Green's function for the unit disk

$$D(0,1) = \{(x_1, x_2)|x_1^2 + x_2^2 < 1\} \subset \mathbb{R}^2.$$

Solution: We use the same idea as for $n \geq 3$. Let $\rho = \sqrt{x_1^2 + x_2^2} < 1$, $\Phi(x)$ be the fundamental solution. We recall that

$$\Phi(x) = -\frac{1}{2\pi} \log(|x|) = \Phi(\rho).$$

We will also use the inversion to invert the singularity out of the disk. Note for $\bar{x} = \frac{x}{|x|^2}$

$$|x||y - \bar{x}| = |y - x|, \; if \; |y| = 1.$$

The function $\Phi(|x|(y - \bar{x}))$ is harmonic in y if $x, y \in D(0,1)$ and

$$\Phi(|x|(y - \bar{x})) = \Phi(x - y), \; for \; y \in \partial D(0,1).$$

Therefore, we have the Green's function for the unit disk

$$G(x,y) = \Phi(x - y) - \Phi(|x|(y - \bar{x})), \; x \neq y \in D(0,1).$$

We now solve the Laplace equation on $D(0,1)$ with boundary data $g(x)$. For this purpose, for $|y| = 1$, we compute

$$\frac{\partial G(y-x)}{\partial \nu} = -\frac{1}{2\pi} \frac{1 - |x|^2}{|x - y|^2}, \; \forall |y| = 1.$$

Therefore, we arrived at the *Poisson's formula*

$$u(x) = \frac{1 - |x|^2}{2\pi} \int_{\partial D(0,1)} \frac{g(y)}{|x - y|^2} \, d\sigma_y. \tag{3.56}$$

We could now use polar coordinate to have another form of the above equation. Let $x_1 = \rho\cos(\theta), \; x_2 = \rho\sin(\theta)$. and $y_1 = \cos(\phi), \; y_2 = \sin(\phi)$, we thus have

$$u(\rho, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - \rho^2) g(\phi)}{1 - 2\rho\cos(\phi - \theta) + \rho^2} \, d\phi. \tag{3.57}$$

If the unit disk is replaced by a general disk $D(0,r)$, (3.56) and (3.57) are replaced by

$$u(x) = \frac{r^2 - |x|^2}{2\pi r} \int_{\partial D(0,r)} \frac{g(y)}{|x - y|^2} \, d\sigma_y. \tag{3.58}$$

$$u(\rho, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(r^2 - \rho^2) g(\phi)}{r^2 - 2r\rho\cos(\phi - \theta) + \rho^2} \, d\phi. \tag{3.59}$$

Example 3.5.2 Find the Green's function for the upper half plane

$$\mathbb{R}^2_+ = \{(x_1, x_2) | x_2 > 0\}.$$

Solution. Similar to subsection 3.3.2, we choose

$$G(x, y) = \Phi(y - x) - \Phi(y - \bar{x})$$

where

$$\bar{x} = (x_1, -x_2), \; for \; x = (x_1, x_2) \in \mathbb{R}^2_+.$$

Therefore

$$G(x, y) = -\frac{1}{2\pi} \log \left(\frac{\sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}}{\sqrt{(y_1 - x_1)^2 + (y_2 + x_2)^2}} \right). \tag{3.60}$$

The corresponding *Poisson's formula* for Laplace equation with boundary data $g(x)$ on $\mathbb{R}^2_+$ is

$$u(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x_2 g(y_1)}{\sqrt{(y_1 - x_1)^2 + x_2^2}} \, dy_1 \tag{3.61}$$

Example 3.5.3 For $a > 0$, $D(0, a)$ is the disk centered at the origin on $\mathbb{R}^2$. Solve the following boundary value problem

$$\begin{cases} -\Delta u = 0, \; in \; D(0, a), \\ u(a, \phi) = g(\phi) = \begin{cases} 1, & 0 < \phi < \pi, \\ 0, & \pi < \phi < 2\pi. \end{cases} \end{cases}$$

Solution: We could use (3.59) to solve this problem:

$$\begin{aligned} u(\rho, \theta) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{(a^2 - \rho^2) g(\phi)}{a^2 - 2a\rho\cos(\phi - \theta) + \rho^2} \, d\phi \\ &= \frac{a^2 - \rho^2}{2\pi} \int_0^{\pi} \frac{1}{(a^2 + \rho^2) - 2a\rho\cos(\phi - \theta)} \, d\phi. \end{aligned}$$

Set $c = a^2 + \rho^2$, $d = -2a\rho$ and $\xi = \phi - \theta$, we reduce the above equation into

$$u(\rho, \theta) = \frac{a^2 - \rho^2}{2\pi} \int_{-\theta}^{\pi - \theta} \frac{1}{c + d\cos(\xi)} \, d\xi.$$

Except for the singular points $\xi = \pm\pi$, we have that

$$F(\xi) = \frac{2}{\sqrt{c^2 - d^2}} \arctan \left(\frac{\sqrt{c^2 - d^2} \tan(\frac{\xi}{2})}{c + d} \right),$$

is the anti-derivative of

$$\frac{1}{c+d\cos(\xi)}.$$

We note that $\phi \in (0,\pi)$ and $\theta \in [0,2\pi]$, therefore, if $\theta \in (0,\pi)$, $\xi \in (-\pi,\pi)$, and so

$$\begin{aligned}
u(\rho,\theta) &= \lim_{\varepsilon\to 0}\left(\frac{a^2-\rho^2}{2\pi}\frac{2}{a^2-\rho^2}\arctan\left(\frac{\sqrt{c^2-d^2}\tan(\frac{\phi-\theta}{2})}{c+d}\right)\right)\Bigg|_{\phi=\varepsilon}^{\phi=\pi-\varepsilon}\\
&= \frac{1}{\pi}\arctan(\frac{a+\rho}{a-\rho}cot(\frac{\theta}{2})) + \frac{1}{\pi}\arctan(\frac{a+\rho}{a-\rho}tan(\frac{\theta}{2}))
\end{aligned}$$

However, if $\theta \in (\pi, 2\pi)$, then $\xi \in (-2\pi, 0)$ which contains $\xi = -\pi$. We have to split the integral at $\xi = -\pi$. Therefore, we have

$$\begin{aligned}
u(\rho,\theta) &= \lim_{\varepsilon\to 0}\left(\frac{1}{\pi}\arctan\left(\frac{a+\rho}{a-\rho}\tan(\frac{\phi-\theta}{2})\right)\right)\Bigg|_{\phi=\varepsilon}^{\phi=\theta-\pi-\varepsilon}\\
&+\lim_{\varepsilon\to 0}\left(\frac{1}{\pi}\arctan\left(\frac{a+\rho}{a-\rho}\tan(\frac{\phi-\theta}{2})\right)\right)\Bigg|_{\phi=\theta-\pi+\varepsilon}^{\phi=\pi-\varepsilon}\\
&= \frac{1}{\pi}\frac{\pi}{2} + \frac{1}{\pi}\arctan(\frac{a+\rho}{a-\rho}tan(\frac{\theta}{2}))\\
&+ \frac{1}{\pi}\arctan(\frac{a+\rho}{a-\rho}cot(\frac{\theta}{2})) + \frac{1}{\pi}\frac{\pi}{2}\\
&= 1 + \frac{1}{\pi}\arctan(\frac{a+\rho}{a-\rho}cot(\frac{\theta}{2})) + \frac{1}{\pi}\arctan(\frac{a+\rho}{a-\rho}tan(\frac{\theta}{2})).
\end{aligned}$$

One easily verify the boundary condition by taking limit $\rho \to a$ in the above two cases.

The above example shows that using Poisson's formula could lead to tedious calculations. We show in the following couple examples some specific tricks in two and three dimensions.

Example 3.5.4 Solve the following Dirichlet problem

$$\begin{cases} -\Delta u = 0, \ in\ D(0,1) \\ u(\rho,\theta) = A\sin^2\theta + B\cos^2\theta, \ on\ \rho = 1, \end{cases}$$

where $x = (x_1, x_2) = (\rho\cos\theta, \rho\sin\theta)$ and A and B are constants.

Solution. It is easy to check that $\rho^n \sin(n\theta)$ and $\rho^n\cos(n\theta)$ are harmonic functions on $\mathbb{R}^2$, and

$$A\sin^2\theta + B\cos^2\theta = \frac{A+B}{2} + \frac{B-A}{2}\cos(2\theta),$$

therefore, we find the solution

$$u(\rho,\theta) = \frac{A+B}{2} + \frac{B-A}{2}\rho^2\cos(2\theta).$$

Example 3.5.5 Find a bounded solution to the following Dirichlet problem outside a unit ball in $\mathbb{R}^3$:

$$\begin{cases} -\Delta u = 0, \quad r > 1, \\ u|_{r=1} = \frac{2}{5+4x_2}, \end{cases}$$

where $r = |x|$.

Solution. We know that the function

$$u(x) = \frac{1}{|x - x_0|}$$

is a harmonic function out of the unit ball if $x_0 \in B(0,1)$. We try to see if we could choose a x_0 so that the above function satisfies the boundary condition.

Since $\frac{2}{5+4x_2} = (\frac{5}{4} + x_2)^{-1}$, we need to find x_0 such that

$$\frac{5}{4} + x_2 = |x - x_0|^2 = 1 - 2x \cdot x_0 + x_0^2$$

which is equivalent to

$$x_{01} = x_{03} = 0, \; x_{02} = -\frac{1}{2}.$$

And such a point is inside the unit ball. Therefore,

$$u(x) = \frac{1}{\sqrt{x_1^2 + (x_2 + \frac{1}{2})^2 + x_3^2}}.$$

Example 3.5.6 Let Ω be the triangle on $\mathbb{R}^2$ with vertices $(-1,0)$, $(1,0)$ and $(0,\sqrt{3})$. Solve the following Dirichlet problem

$$\begin{cases} -\Delta u = 2, \; in \; \Omega \\ u = 0, \; on \; \partial\Omega. \end{cases}$$

Solution. We first observe the equations for the sides of the triangle are

$$y = 0, \; y + \sqrt{3}x - \sqrt{3} = 0, \; y - \sqrt{3}x - \sqrt{3} = 0.$$

We thus guess that the solution has the following form

$$u(x,y) = cy(y + \sqrt{3}x - \sqrt{3})(y - \sqrt{3}x - \sqrt{3})$$

with c the constant to be determined. Clearly, the boundary condition is fulfilled. A direct calculation gives

$$-\Delta u = 4\sqrt{3}c = 2,$$

and so $c = \frac{\sqrt{3}}{6}$ and the solution is

$$u(x,y) = \frac{\sqrt{3}}{6} y(y + \sqrt{3}x - \sqrt{3})(y - \sqrt{3}x - \sqrt{3}).$$

The following example is the famous Hadamard's three circles theorem.

Example 3.5.7 Let D be a annular region on $\mathbb{R}^2$ centered at the origin. The outer circle has radius R_1 and the inner one has radius R_1, $u(x,y)$ is a sub-harmonic function on D. Set

$$M(r) = \max_{x^2+y^2=r^2} u(x,y),\ R_1 < r_1 < r < r_2 < R_2,$$

then

$$M(r) \le \frac{M(r_1)\log(\frac{r_2}{r}) + M(r_2)\log(\frac{r}{r_1})}{\log(\frac{r_2}{r_1})}.$$

Solution. For $r \neq 0$, we define

$$\phi(r) = a + b\log r$$

where a and b are chosen such that

$$\phi(r_1) = M(r_1),\ \phi(r_2) = M(r_2).$$

Therefore, we find

$$\phi(r) = \frac{M(r_1)\log(\frac{r_2}{r}) + M(r_2)\log(\frac{r}{r_1})}{\log(\frac{r_2}{r_1})}.$$

Consider now

$$v(x,y) = u(x,y) - \phi(\sqrt{x^2+y^2}),$$

which satisfies

$$\begin{cases} -\Delta v \le 0,\ if\ r_1 < r < r_2 \\ v \le 0,\ if\ r = r_1\ or\ r = r_2. \end{cases}$$

By the maximum principle, we know that

$$v \le 0,\ if\ r_1 < r < r_2.$$

Therefore,

$$u(x,y) \le \phi(r), if\ r_1 < r < r_2.$$

This implies that

$$M(r) \le \phi(r), if\ r_1 < r < r_2.$$

3.6 Problems

Problem 1. Show that the Laplace operator takes the following form under cylindrical coordination (r, θ, z):

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}.$$

Problem 2. Show that the Laplace operator takes the following form under spherical coordinate (r, θ, ϕ).

$$\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r}) + \frac{1}{r^2 \sin^2\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\frac{\partial u}{\partial \theta}) + \frac{\partial^2 u}{\partial \phi^2}\right].$$

Problem 3. Prove the following functions are harmonic.

- (a) $x^3 - 3xy^2$, and $3x^2y - y^3$.
- (b) $sh(ny)sin(nx)$, $sh(ny)cos(nx)$, $ch(ny)sin(nx)$, and $ch(ny)cos(nx)$.
- (c) $sh(x)(ch(x) + cos(y))^{-1}$ and $sin(y)(ch(x) + cos(y))^{-1}$.

Problem 4. Prove the following functions are harmonic in the polar coordinate.

- (a) $\ln(r)$, and θ.
- (b) $r^n \cos(n\theta)$ and $r^n \sin(n\theta)$.
- (c) $r\ln(r)\cos(\theta) - r\theta\sin(\theta)$ and $r\ln(r)\sin(\theta) + r\theta\cos(\theta)$.

Problem 5. Find the Green's function for the first quadrant of $\mathbb{R}^2$, namely the domain

$$\Omega = \{(x, y) \in \mathbb{R}^2 | x > 0, y > 0\}.$$

Problem 6. Find the Green's function for the upper half ball $B^+(0, r)$ in $\mathbb{R}^3$.

Problem 7. Find the Green's function for the first octant in $\mathbb{R}^3$, namely the domain

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 | x > 0, y > 0, z > 0\}.$$

Problem 8. Find the Green's function for the unit square in $\mathbb{R}^2$, namely the domain

$$\Omega = \{(x, y) \in \mathbb{R}^2 | 1 > x > 0, 1 > y > 0\}.$$

Problem 9. Let $D(0, r)$ is the disk on $\mathbb{R}^2$ with boundary C. For each of the following boundary conditions, find the function u so that it is harmonic on $D(0, r)$.

- (a) $u|_C = A\cos(\phi)$.
- (b) $u|C = A + B\sin(\phi)$.

Problem 10. Solve the following Dirichlet problem

$$\begin{cases} u_{xx} + u_{yy} + u_{zz} = 0, \ x^2 + y^2 + z^2 < 1, \\ u(r, \theta, \phi)|_{r=1} = 3\cos(2\theta) + 1, \end{cases}$$

where (r, θ, ϕ) is the spherical coordinate.

Problem 11. Let B be a unit ball in $\mathbb{R}^n$ $(n \geq 2)$, and u is the smooth solution of the following problem

$$\begin{cases} -\Delta u = f \ in \ B \\ u = g, \ on \ \partial B. \end{cases}$$

Prove that there exists a constant C, depending only on n, such that

$$\max_B |u| \leq C(\max_{\partial B} |g| + \max_{\partial B} |f|).$$

Problem 12. Assume u is harmonic, prove the following statements

- (a) If $\phi : \mathbb{R} \to \mathbb{R}$ is a smooth convex function, then $v = \phi(u)$ is sub-harmonic.
- (b) Prove $v = |Du|^2$ is sub-harmonic.

Problem 13. Use Poisson's formula for the ball to prove

$$r^{n-2} \frac{r - |x|}{(r + |x|)^{n-1}} u(0) \leq u(x) \leq r^{n-2} \frac{r + |x|}{(r - |x|)^{n-1}} u(0),$$

where $u(x)$ is harmonic for $x \in B(0, r) \subset \mathbb{R}^n$ with $n \geq 3$. This is an explicit form of Harnack's inequality.

Problem 14. Let u be the solution of

$$\begin{cases} -\Delta u = 0, \ in \ \mathbb{R}^n_+ \\ u = g \ on \ \partial \mathbb{R}^n_+ \end{cases}$$

given by the Poisson's formula for the half-space. Assume g is bounded and $g(x) = |x|$ for $x \in \partial \mathbb{R}^n_+$, $|x| \leq 1$. Show Du is *not* bounded near $x = 0$.

Problem 15. Let $\Omega_+ \subset \mathbb{R}^n_+$ and $T = \partial \Omega_+ \cap \partial \mathbb{R}^n_+$ is a non-empty open set. Assume $u \in C(\bar{\Omega}_+)$ is harmonic in Ω_+, with $u = 0$ on T. Set

$$v(x) = \begin{cases} u(x_1, \cdots, x_{n-1}, x_n), & x_n > 0 \\ -u(x_1, \cdots, x_{n-1}, -x_n), & x_n < 0. \end{cases}$$

Prove that $v(x)$ is harmonic on $\Omega_+ \cup T \cup \Omega_-$ where Ω_- is the reflection of Ω_+ about $x_n = 0$. This result is called Schwarz reflection theorem.

Problem 16. Using example to show that the maximum principle is not valid for

$$u_{xx} + u_{yy} + cu = 0, \ c > 0.$$

Problem 17. Find the Green's function for the following wedge:

$$\Omega = \{(\rho, \theta, z) : \rho > 0, 0 < \theta < \frac{\pi}{4}, z \in \mathbb{R}\}.$$

Problem 18. Find the Green's function for a domain between two parallel planes:

$$\Omega = \{(x, y, z) : 0 < z < 1\}.$$