Bifurcation Theory

Lecture 5

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The MSM for Bifurcation Analysis: General Systems

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NON-DEFECTIVE BIFURCATIONS OF FINITE-DIMENSIONAL AUTONOMOUS SYSTEMS

- **General finite-dimensional dynamical system:**

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \boldsymbol{\mu}) \quad \mathbf{x} \in \mathbb{R}^N \quad \boldsymbol{\mu} \in \mathbb{R}^M$$

- **Trivial equilibrium path:**

$$\mathbf{F}(\mathbf{0}, \boldsymbol{\mu}) = \mathbf{0} \quad \forall \boldsymbol{\mu} \quad \Rightarrow \quad \mathbf{x}_E = \mathbf{0} \quad \forall \boldsymbol{\mu}$$

- **The eigenvalue problem governs bifurcation from the trivial path:**

$$[\mathbf{J}(\boldsymbol{\mu}_c) - \lambda \mathbf{I}]\mathbf{u} = \mathbf{0}, \qquad \mathbf{J}(\boldsymbol{\mu}_c) := \mathbf{F}_{\mathbf{x}}(\mathbf{0}, \boldsymbol{\mu}_c)$$

Static bifurcation	Dynamic (Hopf) bifurcation
$\lambda = 0, \quad \mathrm{d}\lambda / \mathrm{d}\boldsymbol{\mu} \neq \mathbf{0}$	$\mathrm{Re}[\lambda] = 0, \quad \mathrm{d}(\mathrm{Re}[\lambda]) / \mathrm{d}\boldsymbol{\mu} \neq \mathbf{0}$

■ EXAMPLE: DIVERGENCE-HOPF-HOPF BIFURCATION

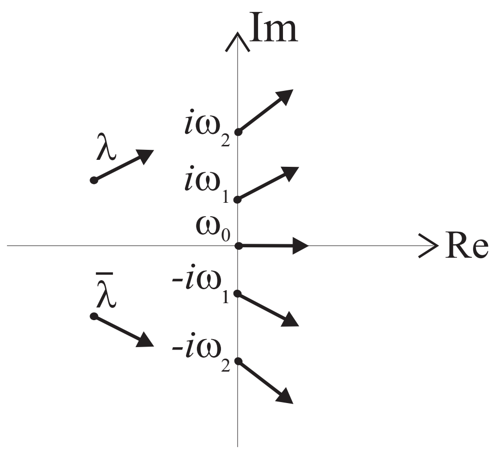
- **Critical eigenvectors (at $\boldsymbol{\mu} = \mathbf{0}$):**

  - **Right:**

$$\mathbf{F}_{\mathbf{x}}^0 \mathbf{u}_0 = \mathbf{0}, \qquad \mathbf{F}_{\mathbf{x}}^0 \mathbf{u}_1 = i\omega_1 \mathbf{u}_1, \qquad \mathbf{F}_{\mathbf{x}}^0 \mathbf{u}_2 = i\omega_2 \mathbf{u}_2$$

  - **Left:**

$$(\mathbf{F}_{\mathbf{x}}^0)^T \mathbf{v}_0 = \mathbf{0}, \quad (\mathbf{F}_{\mathbf{x}}^0)^T \mathbf{v}_1 = -i\omega_1 \mathbf{v}_1, \quad (\mathbf{F}_{\mathbf{x}}^0)^T \mathbf{v}_2 = -i\omega_2 \mathbf{v}_2$$

	**Nonresonant**	$\omega_2 / \omega_1 \neq 1, 2, 3, \cdots$
	Resonant defective	$\omega_2 / \omega_1 = 1$
	Resonant Non-defective	$\omega_2 / \omega_1 = 2, 3, \cdots$

- **Equations expanded around the bifurcation point x=0, μ=0:**

$$\dot{\mathbf{x}} = \mathbf{F}_{\mathbf{x}}^0\mathbf{x} + \frac{1}{2}(\mathbf{F}_{\mathbf{xx}}^0\mathbf{x}^2 + 2\mathbf{F}_{\mathbf{x}\boldsymbol{\mu}}^0\,\mathbf{x}\boldsymbol{\mu}) + \frac{1}{6}(\mathbf{F}_{\mathbf{xxx}}^0\mathbf{x}^3 + 3\mathbf{F}_{\mathbf{xx}\boldsymbol{\mu}}^0\,\mathbf{x}^2\boldsymbol{\mu} + 3\mathbf{F}_{\mathbf{x}\boldsymbol{\mu}\boldsymbol{\mu}}^0\,\mathbf{x}\boldsymbol{\mu}^2) + \cdots$$

- **Rescaling:**

$$\mathbf{x} \to \varepsilon\mathbf{x}, \quad \boldsymbol{\mu} \to \varepsilon\boldsymbol{\mu}$$

- **Rescaled equations:**

$$\dot{\mathbf{x}} = \mathbf{F}_{\mathbf{x}}^0\mathbf{x} + \frac{1}{2}\varepsilon(\mathbf{F}_{\mathbf{xx}}^0\mathbf{x}^2 + 2\mathbf{F}_{\mathbf{x}\boldsymbol{\mu}}^0\,\mathbf{x}\boldsymbol{\mu}) + \frac{1}{6}\varepsilon^2(\mathbf{F}_{\mathbf{xxx}}^0\mathbf{x}^3 + 3\mathbf{F}_{\mathbf{xx}\boldsymbol{\mu}}^0\,\mathbf{x}^2\boldsymbol{\mu} + 3\mathbf{F}_{\mathbf{x}\boldsymbol{\mu}\boldsymbol{\mu}}^0\,\mathbf{x}\boldsymbol{\mu}^2) + \cdots$$

- **Series expansion and independent time-scales:**

$$\mathbf{x}(t;\varepsilon) = \mathbf{x}_0(t_0,t_1,t_2,\cdots) + \varepsilon\mathbf{x}_1(t_0,t_1,t_2,\cdots) + \varepsilon^2\mathbf{x}_2(t_0,t_1,t_2,\cdots) + \cdots$$

$$t_k = \varepsilon^k t, \quad \frac{d}{dt} = d_0 + \varepsilon\, d_1 + \varepsilon^2\, d_2 + \ldots + \varepsilon^k\, d_k, \quad d_k := \frac{\partial}{\partial t_k}$$

- **Perturbation equations:**

$$\varepsilon^0 : (d_0 - \mathbf{F}_{\mathbf{x}}^0)\,\mathbf{x}_0 = \mathbf{0}$$

$$\varepsilon^1 : (d_0 - \mathbf{F}_{\mathbf{x}}^0)\,\mathbf{x}_1 = -d_1\mathbf{x}_0 + \frac{1}{2}\mathbf{F}_{\mathbf{xx}}^0\,\mathbf{x}_0^2 + \mathbf{F}_{\mathbf{x}\boldsymbol{\mu}}^0\,\mathbf{x}_0\boldsymbol{\mu}$$

$$\varepsilon^2 : (d_0 - \mathbf{F}_{\mathbf{x}}^0)\,\mathbf{x}_2 = -d_2\mathbf{x}_0 - d_1\mathbf{x}_1 + \mathbf{F}_{\mathbf{xx}}^0\,\mathbf{x}_0\mathbf{x}_1 + \mathbf{F}_{\mathbf{x}\boldsymbol{\mu}}^0\,\mathbf{x}_1\boldsymbol{\mu} \; +$$

$$+\frac{1}{6}\mathbf{F}_{\mathbf{xxx}}^0\,\mathbf{x}_0^3 + \frac{1}{2}\mathbf{F}_{\mathbf{xx}\boldsymbol{\mu}}^0\,\mathbf{x}_0^2\boldsymbol{\mu} + \frac{1}{2}\mathbf{F}_{\mathbf{x}\boldsymbol{\mu\mu}}^0\,\mathbf{x}_0\boldsymbol{\mu}^2$$

- **Lower-order solution:**

By retaining the *non-decaying* modes only:

$$\mathbf{x}_0 = A_0(t_1, t_2, \ldots)\,\mathbf{u}_0 + A_1(t_1, t_2, \ldots)\,\mathbf{u}_1 e^{i\omega_1 t_0} + A_2(t_1, t_2, \ldots)\,\mathbf{u}_2 e^{i\omega_2 t_0} + c.c.$$

Note: $\mathbf{x}_0$ belongs to the 5-dimensional central subspace of the Jacobian $\mathbf{F}_{\mathbf{x}}^0$

- **ε-order:**

$$(d_0 - \mathbf{F}_{\mathbf{x}}^0)\,\mathbf{x}_1 = (\mathbf{f}_0 - 2d_1 A_0 \mathbf{u}_0) + (\mathbf{f}_1 - 2d_1 A_1 \mathbf{u}_1)\, e^{i\omega_1 t_0} + (\mathbf{f}_2 - 2d_1 A_2 \mathbf{u}_2)\, e^{i\omega_2 t_0}$$
$$+\mathbf{f}_{20}\, e^{2i\omega_1 t_0} + \mathbf{f}_{02}\, e^{2i\omega_2 t_0} + \mathbf{f}_{11}\, e^{i(\omega_1+\omega_2)t_0} + \mathbf{f}_{\bar{1}1}\, e^{i(\omega_2-\omega_1)t_0} + c.c.$$

where:

$$\mathbf{f}_0 := 2A_0^2 \mathbf{F}_{\mathbf{xx}}^0 \mathbf{u}_0^2 + A_1 \bar{A}_1 \mathbf{F}_{\mathbf{xx}}^0 \mathbf{u}_1 \bar{\mathbf{u}}_1 + A_2 \bar{A}_2 \mathbf{F}_{\mathbf{xx}}^0 \mathbf{u}_2 \bar{\mathbf{u}}_2 + 2A_0 \mathbf{F}_{\mathbf{x}\boldsymbol{\mu}}^0 \mathbf{u}_0 \boldsymbol{\mu}_1$$

$$\mathbf{f}_1 := 4A_0 A_1 \mathbf{F}_{\mathbf{xx}}^0 \mathbf{u}_0 \mathbf{u}_1 + 2A_1 \mathbf{F}_{\mathbf{x}\boldsymbol{\mu}}^0 \mathbf{u}_1 \boldsymbol{\mu}_1, \quad \mathbf{f}_2 := 4A_0 A_2 \mathbf{F}_{\mathbf{xx}}^0 \mathbf{u}_0 \mathbf{u}_2 + 2A_2 \mathbf{F}_{\mathbf{x}\boldsymbol{\mu}}^0 \mathbf{u}_2 \boldsymbol{\mu}_1$$

$$\mathbf{f}_{20} := A_1^2 \mathbf{F}_{\mathbf{xx}}^0 \mathbf{u}_1^2, \quad \mathbf{f}_{02} := A_2^2 \mathbf{F}_{\mathbf{xx}}^0 \mathbf{u}_2^2, \quad \mathbf{f}_{11} := 2A_1 A_2 \mathbf{F}_{\mathbf{xx}}^0 \mathbf{u}_1 \mathbf{u}_2, \quad \mathbf{f}_{\bar{1}1} := 2\bar{A}_1 A_2 \mathbf{F}_{\mathbf{xx}}^0 \bar{\mathbf{u}}_1 \mathbf{u}_2$$

- **Removing secular terms:**

$$(d_0 - \mathbf{F}_{\mathbf{x}}^0)\,\mathbf{x}_1 = (\mathbf{f}_0 - 2d_1A_0\mathbf{u}_0) + (\mathbf{f}_1 - 2d_1A_1\mathbf{u}_1)\,e^{i\omega_1 t_0} + (\mathbf{f}_2 - 2d_1A_2\mathbf{u}_2)\,e^{i\omega_2 t_0}$$
$$+\mathbf{f}_{20}\,e^{2i\omega_1 t_0} + \mathbf{f}_{02}\,e^{2i\omega_2 t_0} + \mathbf{f}_{11}\,e^{i(\omega_1+\omega_2)t_0} + \mathbf{f}_{\bar{1}1}\,e^{i(\omega_2-\omega_1)t_0} + c.c.$$

To avoid secular terms, the resonant excitations of frequencies $\Omega = (0, \omega_1, \omega_2)$ *must be made orthogonal to the* ***left eigenvectors*** *of equal frequency*. We distinguish:

- ***non-resonant*** **systems:**

$$d_1 A_0 = \frac{1}{2}\mathbf{v}_0^T\mathbf{f}_0\,, \quad d_1A_1 = \frac{1}{2}\mathbf{v}_1^H\mathbf{f}_1, \quad d_1A_2 = \frac{1}{2}\mathbf{v}_2^H\mathbf{f}_2$$

- ***resonant*** **systems,** $\omega_2 = 2\omega_1$ **:**

$$d_1 A_0 = \frac{1}{2}\mathbf{v}_0^T\mathbf{f}_0\,, \quad d_1A_1 = \frac{1}{2}\mathbf{v}_1^H(\mathbf{f}_1 + \mathbf{f}_{\bar{1}1}), \quad d_1A_2 = \frac{1}{2}\mathbf{v}_2^H(\mathbf{f}_2 + \mathbf{f}_{20})$$

- **By proceeding at higher-orders:**

  By solving the perturbation equations (with complementary solutions neglected) and removing secular terms, we obtain equations of the following forms:

$$\varepsilon^1 : d_1\mathbf{A} = L_1(\mathbf{A}^2, \mathbf{A}\boldsymbol{\mu})$$

$$\varepsilon^2 : d_2\mathbf{A} = L_2(\mathbf{A}^3, \mathbf{A}^2\boldsymbol{\mu})$$

$$\cdots\cdots\cdots$$

  where $\mathbf{A} = (A_0, A_1, A_2)$.

- **Amplitude equations, recombined:**

  Since $d\mathbf{A}/dt = \varepsilon\, d_1\mathbf{A} + \varepsilon^2\, d_2\mathbf{A} + \cdots$, by reabsorbing ε :

$$\dot{\mathbf{A}} = \mathbf{L}(\mathbf{A}^3, \mathbf{A}^2, \mathbf{A}^2\boldsymbol{\mu}, \mathbf{A}\boldsymbol{\mu}^2)$$

- **In real form:**

$$A_0 = \frac{1}{2} a_0, \quad A_1 = \frac{1}{2} a_1 \, \mathrm{e}^{i\theta_1}, \quad A_2 = \frac{1}{2} a_2 \, \mathrm{e}^{i\theta_2}$$

  - **for *non-resonant* systems:**

$$\dot{\mathbf{a}} = f(\mathbf{a}, \mu) \qquad \mathbf{a} := \left(a_0, a_1, a_2\right)^T$$

$$\dot{\boldsymbol{\theta}} = g(\mathbf{a}, \mu) \qquad \boldsymbol{\theta} := \left(\theta_1, \theta_2\right)^T$$

  - **for *resonant* systems,** $\omega_2 = m\omega_1$ **:**

$$\begin{cases} \dot{\mathbf{a}} = \mathbf{f}(\mathbf{a}, \gamma; \mu) \\ \dot{\gamma} = g(\mathbf{a}, \gamma; \mu), \qquad \gamma := m\theta_1 - \theta_2, \quad m = 2, 3, \ldots \end{cases}$$

These equations are bifurcation equations reduced to the Center Manifold, *naturally* appearing in their Normal Form.

- **Steady motions of *non-resonant* systems:**

By letting $\dot{\mathbf{a}} = \mathbf{0}$ we obtain:

$$\mathbf{0} = f(\mathbf{a}, \mu) \quad \Rightarrow \mathbf{a} = \mathbf{a}_s$$

$$\dot{\boldsymbol{\theta}} = g(\mathbf{a}_s, \mu) \quad \Rightarrow \theta_k = \nu_k t + \varphi_k \quad \Rightarrow \Omega_k = \omega_k + \nu_k$$

They are **quasi-periodic** motions of the original system, occurring around a *non-trivial* equilibrium position:

$$\mathbf{x} = a_0 \mathbf{u}_0 + a_1 \left(\mathrm{Re}[\mathbf{u}_1] \cos(\Omega_1 t + \varphi_1) - \mathrm{Im}[\mathbf{u}_1] \sin(\Omega_1 t + \varphi_1) \right)$$
$$+ a_2 \left(\mathrm{Re}[\mathbf{u}_2] \cos(\Omega_2 t + \varphi_2) - \mathrm{Im}[\mathbf{u}_2] \sin(\Omega_2 t + \varphi_2) \right) + h.o.t.$$

- **Stability:**

Stability is governed by the variational equation:

$$\delta \dot{\mathbf{a}} = \mathbf{f}_{\mathbf{a}}^{s} \delta \mathbf{a}$$

- **Steady motions of *resonant* systems:**

By letting $\dot{\mathbf{a}} = \mathbf{0}$, $\dot{\gamma} = 0$ we obtain:

$$\mathbf{0} = f(\mathbf{a}, \gamma, \mu)$$
$$\dot{\gamma} = g(\mathbf{a}, \gamma, \mu) \Rightarrow \mathbf{a} = \mathbf{a}_s, \gamma = \gamma_s \Rightarrow m\theta_2 - \theta_1 = \gamma_s \Rightarrow \Omega_2 = m\Omega_1$$

They are **periodic** motions of the original system:

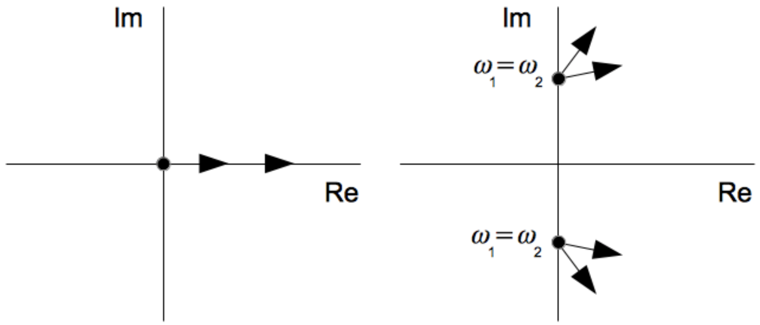
$$\mathbf{x} = a_0\mathbf{u}_0 + a_1\left(\mathrm{Re}[\mathbf{u}_1]\cos(\Omega_1 t + \varphi_1) - \mathrm{Im}[\mathbf{u}_1]\sin(\Omega_1 t + \varphi_1)\right)$$
$$+a_2\left(\mathrm{Re}[\mathbf{u}_2]\cos(m\Omega_1 t + \varphi_2) - \mathrm{Im}[\mathbf{u}_2]\sin(m\Omega_1 t + \varphi_2)\right) + h.o.t.$$

- **Stability:**

$$\begin{pmatrix} \delta\dot{\mathbf{a}} \\ \delta\dot{\gamma} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_{\mathbf{a}}^s & \mathbf{f}_{\gamma}^s \\ g_{\mathbf{a}}^s & g_{\gamma}^s \end{pmatrix} \begin{pmatrix} \delta\mathbf{a} \\ \delta\gamma \end{pmatrix}$$

DEFECTIVE BIFURCATIONS

Two (or more) critical eigenvalues coalesce at the bifurcation:



$$\mathbf{J}_0 = \left(\begin{array}{cc|c} 0 & 1 & \\ & 0 & \\ \hline & & \textbf{stable} \end{array}\right), \qquad \mathbf{J}_0 = \left(\begin{array}{cc|cc|c} i\omega & 1 & & & \\ & i\omega & & & \\ \hline & & -i\omega & 1 & \\ & & & -i\omega & \\ \hline & & & & \textbf{stable} \end{array}\right)$$

The proper eigenvectors are incomplete (they do not span the whole space).

Generalized Eigenvectors $(\mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_n)$ forming a Jordan chain:

$$(\mathbf{J}_0 - \lambda_0 \mathbf{I})\mathbf{u}_1 = \mathbf{0}$$

$$\ldots$$

$$(\mathbf{J}_0 - \lambda_0 \mathbf{I})\mathbf{u}_k = \mathbf{u}_{k-1} \qquad k = 2, 3, \ldots, n-1$$

Orthogonality to the left eigenvector:

$$\mathbf{v}^H \mathbf{u}_k = 0 \qquad k = 1, 2, \ldots, n-1$$

$$\mathbf{v}^H \mathbf{u}_n \neq 0$$

Failure of the standard MSM (based on integer expansions):

$$\mathbf{v}^H \mathbf{f}_{res} = 0 \quad \Rightarrow \quad \underbrace{\mathbf{v}^H \mathbf{u}_1}_{=0!} d_k \mathbf{A} = \mathbf{F}_k(\mathbf{A}; \boldsymbol{\mu}) \quad k = 1, 2, \ldots$$

Fractional (Puisex) series expansions algorithm.
Example: Double-zero bifurcation

Fractional time-scales:

$$t_0 = t, \qquad t_1 = \varepsilon^{1/2} t, \qquad t_2 = \varepsilon^{2/2} t, \qquad \dots$$

$$d/dt = \sum_{k=0}^{\infty} \varepsilon^{k/2} d_{k/2}, \qquad d_{k/2} := \partial / \partial t_{k/2}$$

Expansion:

$$\mathbf{x} = \sum_{k=0}^{\infty} \varepsilon^{k/2} \mathbf{x}_{k/2}$$

Perturbation equations:

$$\varepsilon^0 : \quad \left(d_0 - \mathbf{J}_0\right)\mathbf{x}_0 = \mathbf{0}$$

$$\varepsilon^{1/2} : \quad \left(d_0 - \mathbf{J}_0\right)\mathbf{x}_{1/2} = -d_{1/2}\mathbf{x}_0$$

$$\varepsilon^1 : \quad \left(d_0 - \mathbf{J}_0\right)\mathbf{x}_1 = -\left(d_1\mathbf{x}_0 + d_{1/2}\mathbf{x}_{1/2}\right) + \boldsymbol{\mu}\mathbf{J}_1\mathbf{x}_0 + \mathbf{n}_2\left(\mathbf{x}_0, \mathbf{x}_0\right)$$

Solution:

$$\varepsilon^0: \qquad \mathbf{x}_0 = a\left(t_{1/2}, t_1, \ldots\right)\mathbf{u}_0$$

$$\varepsilon^{1/2}: \quad \textit{solvability not required}! \qquad \mathbf{x}_{1/2} = -\left(d_{1/2}a\right)\mathbf{u}_2$$

$$\varepsilon^1: \qquad \textit{solvability provides}: \qquad d_{1/2}^2 a = F_{1/2}\left(a\ ;\boldsymbol{\mu}\right)$$

$$\varepsilon^{>1}: \qquad \ldots$$

Bifurcation equations:

$$\ddot{a} = F\left(a, \dot{a}\ ;\boldsymbol{\mu}\right)$$

The motion takes place on a 2-dimensional manifold.