Bifurcation Theory

Lecture 4

a.y. 2013/14

The MSM for Bifurcation Analysis: Sample Non-Defective Systems

ANGELO LUONGO

STATIC BIFURCATIONS

■ **A two-dimensional system, undergoing a simple divergence bifurcation**

A system already analyzed by CMM, with an imperfection η added:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} \mu & 0 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} xy + cx^3 + \eta \\ bx^2 \end{pmatrix}$$

- Rescaling:

$$(x, y) \to (\varepsilon x, \varepsilon y), \quad \mu \to \varepsilon^2 \mu, \quad \eta \to \varepsilon^3 \eta$$

The equations become:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \varepsilon \begin{pmatrix} xy \\ bx^2 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} \mu x + cx^3 + \eta \\ 0 \end{pmatrix}$$

- Series expansions:

$$\begin{pmatrix} x(t;\varepsilon) \\ y(t;\varepsilon) \end{pmatrix} = \begin{pmatrix} x_0(t_0,t_1,t_2,\cdots) \\ y_0(t_0,t_1,t_2,\cdots) \end{pmatrix} + \varepsilon \begin{pmatrix} x_1(t_0,t_1,t_2,\cdots) \\ y_1(t_0,t_1,t_2,\cdots) \end{pmatrix} + \varepsilon^2 \begin{pmatrix} x_2(t_0,t_1,t_2,\cdots) \\ y_2(t_0,t_1,t_2,\cdots) \end{pmatrix} + \cdots$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \mathrm{d}_0 + \varepsilon\,\mathrm{d}_1 + \varepsilon^2\,\mathrm{d}_2 + \cdots, \quad \mathrm{d}_k := \partial/\partial t_k, \quad t_k := \varepsilon^k t_k$$

- Perturbation equations:

$$\varepsilon^0 : \begin{cases} \mathrm{d}_0\, x_0 = 0 \\ \mathrm{d}_0\, y_0 + y_0 = 0 \end{cases}$$

$$\varepsilon^1 : \begin{cases} \mathrm{d}_0\, x_1 = -\mathrm{d}_1 x_0 + x_0 y_0 \\ \mathrm{d}_0\, y_1 + y_1 = -\mathrm{d}_1 y_0 + b x_0^2 \end{cases}$$

$$\varepsilon^2 : \begin{cases} \mathrm{d}_0\, x_2 = -\mathrm{d}_2 x_0 - \mathrm{d}_1 x_1 + (x_1 y_0 + x_0 y_1) + \mu x_0 + c x_0^3 + \eta \\ \mathrm{d}_0\, y_2 + y_2 = -\mathrm{d}_2 y_0 - \mathrm{d}_1 y_1 + 2b x_0 x_1 \end{cases}$$

………

- Generating solution:

$$\begin{cases} x_0 = a(t_1, t_2) \\ y_0 = k(t_1, t_2)\,\mathrm{e}^{-t_0} \end{cases}$$

By ignoring transient motions, the steady contribution only is retained:

$$\begin{cases} x_0 = a(t_1, t_2) \\ y_0 = 0 \end{cases}$$

- ❑ **Note:** the passive variable *y does not* enter the generating solution.

- ε-order:

  - equations:

$$\begin{cases} d_0 x_1 = -d_1 a \\ d_0 y_1 + y_1 = b a^2 \end{cases}$$

  - elimination of secular terms:

$$d_1 a = 0$$

  - solution:
    By omitting the complementary solutions:

$$\begin{cases} x_1 = 0 \\ y_1 = b a^2 \end{cases}$$

- **Note:** the link between passive and active coordinates is established at this order.

- ε^2-order:
  - equations:

$$\begin{cases} \mathrm{d}_0\, x_2 = -\mathrm{d}_2 a + \mu a + (b+c)a^3 + \eta \\ \mathrm{d}_0\, y_2 + y_2 = 0 \end{cases}$$

  - elimination of secular terms:

$$\mathrm{d}_2 a = \mu a + (b+c)a^3 + \eta$$

- By coming back to the original, not rescaled, variables, through:

$$\varepsilon a \to a, \quad \varepsilon^2 \mu \to \mu, \quad \varepsilon^3 \eta \to \eta, \quad \varepsilon^2\, \mathrm{d}_2 \to \mathrm{D}$$

the *bifurcation equation* follows:

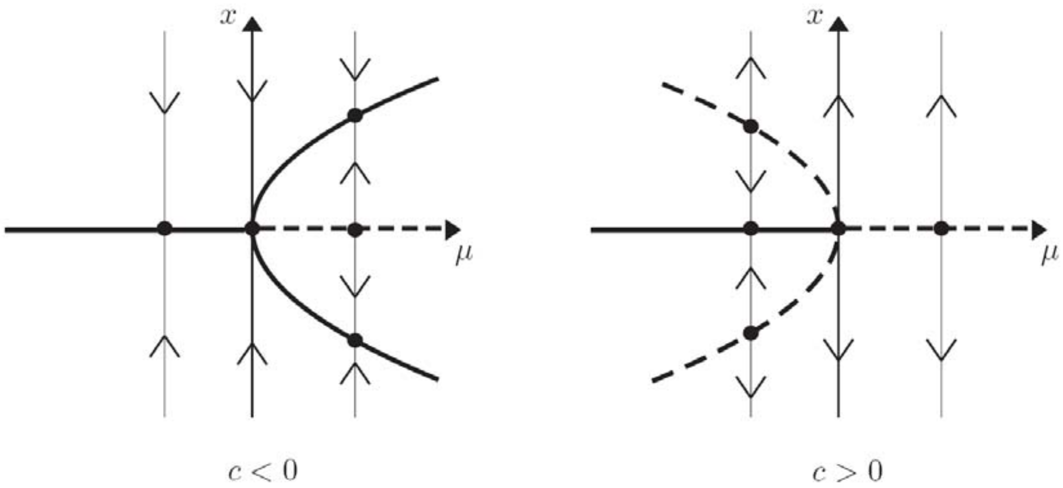
$$\dot{a} = \mu a + (b+c)a^3 + \eta$$

(coincident) with that furnished by the CMM, with the imperfection added.

- Bifurcation diagram for the perfect system $\eta = 0$:

$$\dot{x} = \mu x + cx^3 \text{ (quantities renamed)}$$

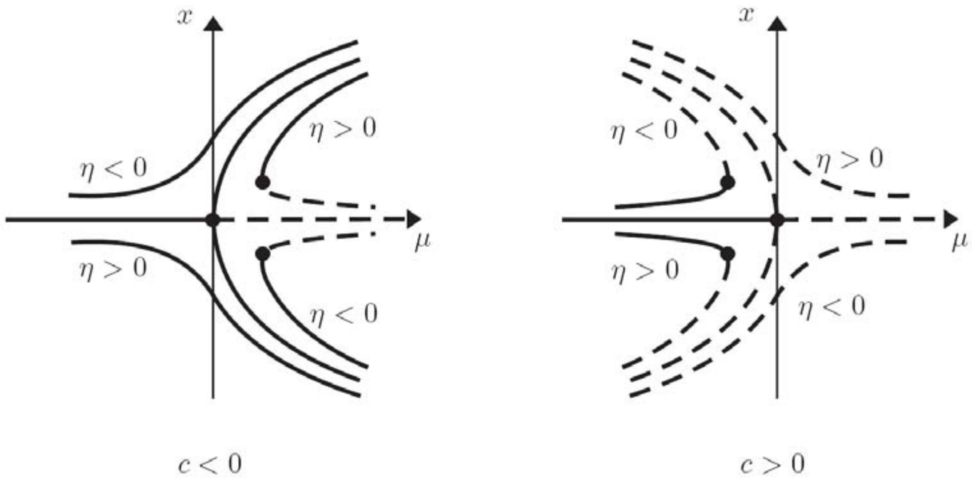
One *or* three equilibria exist at the same μ: $x_T = 0 \; \forall \mu$ and $x_{NT} = \pm\sqrt{-\mu / c}$ for $\mu/c<0$. This is a *fork bifurcation*, *super-critical* if $c<0$, *sub-critical* if $c>0$.



❑ **Note:** An *exchange of stability* occurs at the bifurcation point.

- Bifurcation diagram for the imperfect system $\eta \neq 0$:

$$\dot{x} = \mu x + cx^3 + \eta \text{ (quantities renamed)}$$



:

- o The branch point is destroyed by imperfections, and saddle-node bifurcation points appear; the fork bifurcation *is structurally unstable*
- o In the sub-critical case, imperfections of both signs *reduce* the maximum stable value of μ.
- o In the super-critical case, imperfections have non-catastrophic character.

■ A three-dimensional system, undergoing a double divergence bifurcation

We study a codimension-2 static bifurcation:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} xz + c_1 x^3 + \eta \\ yz + c_2 y^3 + \eta \\ b_1 x^2 + b_2 y^2 \end{pmatrix}$$

Here, the Jacobian **J** admits the (semi-simple) double eigenvalue λ=0 at

$\boldsymbol{\mu}_c=(\mu_c,\nu_c)=(0,0)$. In the CMM view, $\mathbf{x}_c = (x, y)$, $\mathbf{x}_s = (z)$.

- Rescaling:

  After the rescaling $(x,y,z)\to(\varepsilon x,\varepsilon y,\varepsilon z),\ \mu\to\varepsilon^2\mu,\nu\to\varepsilon^2\nu,\ \eta\to\varepsilon^3\eta$ the equations read:

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&-1\end{bmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}+\varepsilon\begin{pmatrix}xz\\ yz\\ b_1x^2+b_2y^2\end{pmatrix}+\varepsilon^2\begin{pmatrix}\mu x+c_1x^3+\eta\\ \nu y+c_2y^3+\eta\\ 0\end{pmatrix}$$

- Series expansions:

$$\begin{pmatrix}x(t;\varepsilon)\\ y(t;\varepsilon)\\ z(t;\varepsilon)\end{pmatrix}=\begin{pmatrix}x_0(t_0,t_1,t_2,\cdots)\\ y_0(t_0,t_1,t_2,\cdots)\\ z_0(t_0,t_1,t_2,\cdots)\end{pmatrix}+\varepsilon\begin{pmatrix}x_1(t_0,t_1,t_2,\cdots)\\ y_1(t_0,t_1,t_2,\cdots)\\ z_1(t_0,t_1,t_2,\cdots)\end{pmatrix}+\varepsilon^2\begin{pmatrix}x_2(t_0,t_1,t_2,\cdots)\\ y_2(t_0,t_1,t_2,\cdots)\\ z_2(t_0,t_1,t_2,\cdots)\end{pmatrix}+\cdots$$

$$\frac{\mathrm{d}}{\mathrm{d}t}=\mathrm{d}_0+\varepsilon\,\mathrm{d}_1+\varepsilon^2\,\mathrm{d}_2+\cdots,\quad \mathrm{d}_k:=\partial/\partial t_k,\quad t_k:=\varepsilon^k t_k$$

- Perturbation equations:

$$\varepsilon^0:\begin{cases} d_0\, x_0 = 0 \\ d_0\, y_0 = 0 \\ d_0\, z_0 + z_0 = 0 \end{cases}$$

$$\varepsilon^1:\begin{cases} d_0\, x_1 = -d_1 x_0 + x_0 z_0 \\ d_0\, y_1 = -d_1 y_0 + y_0 z_0 \\ d_0\, z_1 + z_1 = -d_1 z_0 + b_1 x_0^2 + b_2 y_0^2 \end{cases}$$

$$\varepsilon^2:\begin{cases} d_0\, x_2 = -d_2 x_0 - d_1 x_1 + (x_1 z_0 + x_0 z_1) + \mu x_0 + c_1 x_0^3 + \eta \\ d_0\, y_2 = -d_2 y_0 - d_1 y_1 + (y_1 z_0 + y_0 z_1) + \nu y_0 + c_2 y_0^3 + \eta \\ d_0\, z_2 + z_2 = -d_2 z_0 - d_1 z_1 + 2b_1 x_0 x_1 + 2b_2 y_0 y_1 \end{cases}$$

………

- Generating solution:

$$\begin{cases} x_0 = a_1(t_1, t_2) \\ y_0 = a_2(t_1, t_2) \\ z_0 = 0 \end{cases}$$

- ε-order:
  - equations:

$$\begin{cases} \mathrm{d}_0\, x_1 = -\mathrm{d}_1 a_1 \\ \mathrm{d}_0\, y_1 = -\mathrm{d}_1 a_2 \\ \mathrm{d}_0\, z_1 + z_1 = b_1 a_1^2 + b_2 a_2^2 \end{cases}$$

  - Secular terms:

$$\mathrm{d}_1 a_1 = 0, \quad \mathrm{d}_1 a_2 = 0$$

  - solution:

$$\begin{cases} x_1 = 0 \\ y_1 = 0 \\ z_1 = b_1 a_1^2 + b_2 a_2^2 \end{cases}$$

- ε^2-order:
  - equations:

$$\begin{cases} d_0 x_2 = -d_2 a_1 + \mu a_1 + (b_1 + c_1) a_1^3 + b_2 a_1 a_2^2 + \eta \\ d_0 y_2 = -d_2 a_2 + \nu a_2 + b_1 a_1^2 a_2 + (b_2 + c_2) a_2^3 + \eta \\ d_0 z_2 + z_2 = 0 \end{cases}$$

  - elimination of secular terms:

$$\begin{cases} d_2 a_1 = \mu a_1 + (b_1 + c_1) a_1^3 + b_2 a_1 a_2^2 + \eta \\ d_2 a_2 = \nu a_2 + b_1 a_1^2 a_2 + (b_2 + c_2) a_2^3 + \eta \end{cases}$$

- Bifurcation equations:

$$\begin{cases} \dot{a}_1 = a_1[\mu + (b_1 + c_1) a_1^2 + b_2 a_2^2] + \eta \\ \dot{a}_2 = a_2[\nu + b_1 a_1^2 + (b_2 + c_2) a_2^2] + \eta \end{cases}$$

• Steady-state solutions for the perfect (η=0) system::

$$(T)\quad : \quad a_1 = 0, a_2 = 0, \quad \forall(\mu,\nu) \quad \text{(Trivial)}$$

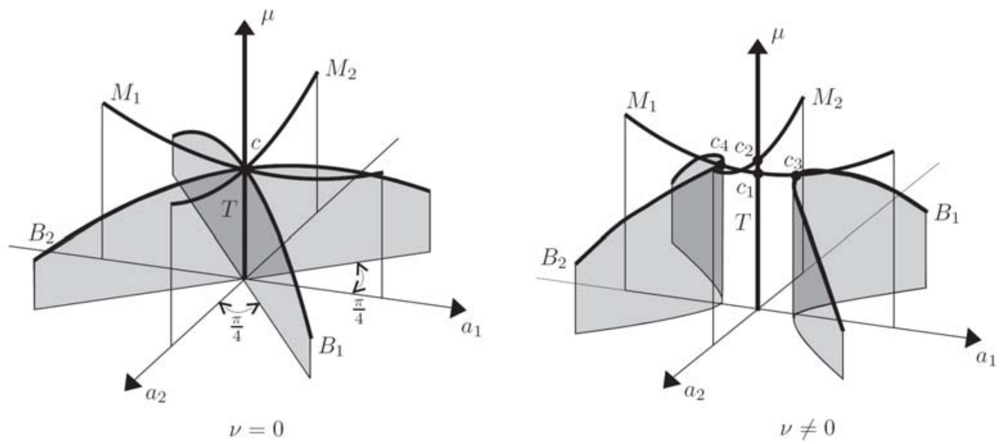
$$(M_1): \quad a_1^2 > 0, a_2 = 0 \quad \text{(Mono-modal)}$$

$$(M_2): \quad a_1 = 0, a_2^2 > 0 \quad \text{(Mono-modal)}$$

$$(B_{1,2}): \quad a_1^2 > 0,\ a_2^2 > 0 \quad \text{(Bi-modal)}$$

Solutions (M_1), (M_2),(B) exist only in a sector of the (μ,ν)-parameter plane. In some sectors more solution can be in competition.

➢Example:



SIMPLE-HOPF BIFURCATION

EXAMPLE: TWO RAYLEIGH-DUFFING OSCILLATORS, ONE STABLE, THE OTHER UNSTABLE

$$\begin{cases} \ddot{x} - \mu\dot{x} + \omega_1^2 x + b_1\dot{x}^3 + cx^3 - b_0(\dot{y} - \dot{x})^3 = 0 \\ \ddot{y} + \xi\dot{y} + \omega_2^2 y + b_2\dot{y}^3 + cy^3 + b_0(\dot{y} - \dot{x})^3 = 0 \end{cases}$$

with $\xi > 0$.

- Rescaling:

$$\mu \to \varepsilon\mu \quad , \quad (x, y) \to (\varepsilon^{1/2} x, \varepsilon^{1/2} y)$$

- Series expansions:

$$\begin{pmatrix} x(t;\varepsilon) \\ y(t;\varepsilon) \end{pmatrix} = \begin{pmatrix} x_0(t_0,t_1,t_2,\cdots) \\ y_0(t_0,t_1,t_2,\cdots) \end{pmatrix} + \varepsilon \begin{pmatrix} x_1(t_0,t_1,t_2,\cdots) \\ y_1(t_0,t_1,t_2,\cdots) \end{pmatrix} + \varepsilon^2 \begin{pmatrix} x_2(t_0,t_1,t_2,\cdots) \\ y_2(t_0,t_1,t_2,\cdots) \end{pmatrix} + \cdots$$

where $t_k \coloneqq \varepsilon^k t_k$.

- Chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}t} = \mathrm{d}_0 + \varepsilon\,\mathrm{d}_1 + \varepsilon^2\,\mathrm{d}_2 + \cdots, \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2} = \mathrm{d}_0^2 + 2\varepsilon\,\mathrm{d}_0\,\mathrm{d}_1 + \varepsilon^2\left(\mathrm{d}_1^2 + 2\,\mathrm{d}_0\,\mathrm{d}_2\right) + \cdots$$

where $\mathrm{d}_k \coloneqq \partial / \partial t_k$.

- Perturbation equations:

$$\varepsilon^0:\begin{cases} \mathrm{d}_0^2 x_0 + \omega_1^2 x_0 = 0 \\ \mathrm{d}_0^2 y_0 + \xi\,\mathrm{d}_0 y_0 + \omega_2^2 y_0 = 0 \end{cases}$$

$$\varepsilon^1:\begin{cases} \mathrm{d}_0^2 x_1 + \omega_1^2 x_1 = -2\,\mathrm{d}_0\mathrm{d}_1 x_0 - b_1(\mathrm{d}_0 x_0)^3 - c x_0^3 + b_0(\mathrm{d}_0 y_0 - \mathrm{d}_0 x_0)^3 + \mu\,\mathrm{d}_0 x_0 \\ \mathrm{d}_0^2 y_1 + \xi\,\mathrm{d}_0 y_1 + \omega_2^2 y_1 = -2\,\mathrm{d}_0\mathrm{d}_1 y_0 - b_2(\mathrm{d}_0 y_0)^3 - c y_0^3 - b_0(\mathrm{d}_0 y_0 - \mathrm{d}_0 x_0)^3 - \xi\,\mathrm{d}_1 y_0 \end{cases}$$

$$\varepsilon^2:\begin{cases} \mathrm{d}_0^2 x_2 + \omega_1^2 x_2 = -(2\,\mathrm{d}_0\mathrm{d}_2 x_0 + \mathrm{d}_1^2 x_0 + 2\,\mathrm{d}_0\mathrm{d}_1 x_1) - 3b_1(\mathrm{d}_0 x_0)^2(\mathrm{d}_1 x_0 + \mathrm{d}_0 x_1) \\ \qquad -3c x_0^2 x_1 + 3b_0(\mathrm{d}_0 y_0 - \mathrm{d}_0 x_0)^2(\mathrm{d}_0 y_1 + \mathrm{d}_1 y_0 - \mathrm{d}_0 x_1 - \mathrm{d}_1 x_0) \\ \qquad +\mu(\mathrm{d}_1 x_0 + \mathrm{d}_0 x_1) \\ \mathrm{d}_0^2 y_2 + \xi\,\mathrm{d}_0 y_2 + \omega_2^2 y_2 = -(2\,\mathrm{d}_0\mathrm{d}_2 y_0 + \mathrm{d}_1^2 y_0 + 2\,\mathrm{d}_0\mathrm{d}_1 y_1) - 3b_2(\mathrm{d}_0 y_0)^2(\mathrm{d}_1 y_0 + \mathrm{d}_0 y_1) \\ \qquad -3c y_0^2 y_1 - 3b_0(\mathrm{d}_0 y_0 - \mathrm{d}_0 x_0)^2(\mathrm{d}_0 y_1 + \mathrm{d}_1 y_0 - \mathrm{d}_0 x_1 - \mathrm{d}_1 x_0) \\ \qquad -\xi(\mathrm{d}_2 y_0 + \mathrm{d}_1 y_1) \end{cases}$$

- Generating solution:

$$x_0 = A(t_1, t_2)\,\mathrm{e}^{i\omega_1 t_0} + c.c., \quad y_0 = 0$$

since the *y*-oscillator is damped. Therefore: *x* = *active coordinate*, and *y*= *passive coordinate*.

- ε -order:

  - equations:

$$\begin{cases} \mathrm{d}_0^2\, x_1 + \omega_1^2 x_1 = f_{1,1}\,\mathrm{e}^{i\omega_1 t_0} + f_{1,3}\,\mathrm{e}^{3i\omega_1 t_0} + c.c. \\ \mathrm{d}_0^2\, y_1 + \xi\,\mathrm{d}_0 y_1 + \omega_2^2 y_1 = f_{2,1}\,\mathrm{e}^{i\omega_1 t_0} + f_{2,3}\,\mathrm{e}^{3i\omega_1 t_0} + c.c. \end{cases}$$

where:

$$f_{1,1} := -2i\omega_1\,\mathrm{d}_1\,A + i\mu\omega_1 A - 3[c + i(b_0 + b_1)\omega_1^3]A^2\bar{A},$$

$$f_{1,3} := [-c + i(b_0 + b_1)\omega_1^3]A^3$$

$$f_{2,1} := 3ib_0\omega_1^3 A^2\bar{A}, \quad f_{2,3} := -ib_0\omega_1^3 A^3$$

➢ elimination of resonant terms requires $f_{1,1} = 0$, from which:

$$\mathrm{d}_1 A = \frac{1}{2}\mu A + \frac{3}{2}[i\frac{c}{\omega_1} - (b_0 + b_1)\omega_1^2]A^2\overline{A}$$

The *y*-equation, does not require any additional condition.

➢ Solution:

$$x_1 = -\frac{f_{1,3}}{8\omega_1^2}\mathrm{e}^{3i\omega_1 t_0} + c.c.,$$

$$y_1 = \frac{f_{2,1}}{\omega_2^2 - \omega_1^2 + i\xi\omega_1}\mathrm{e}^{i\omega_1 t_0} + \frac{f_{2,3}}{\omega_2^2 - 9\omega_1^2 + 3i\xi\omega_1}\mathrm{e}^{3i\omega_1 t_0} + c.c.$$

❑ **Note:** *Only the particular solutions have been considered*, since the complementary *x*-solution repeats the generating one, and the complementary *y*-solution decays in time.

- ε^2 -order:

  - equations:

$$\mathrm{d}_0^2 x_2 + \omega_1^2 x_2 = -(2\,\mathrm{d}_0\mathrm{d}_2 x_0 + \mathrm{d}_1^2 x_0 + 2\,\mathrm{d}_0\mathrm{d}_1 x_1) - 3b_1(\mathrm{d}_0 x_0)^2(\mathrm{d}_1 x_0 + \mathrm{d}_0 x_1)$$
$$-3c x_0^2 x_1 + 3b_0(\mathrm{d}_0 x_0)^2(\mathrm{d}_0 y_1 - \mathrm{d}_0 x_1 - \mathrm{d}_1 x_0) + \mu(\mathrm{d}_1 x_0 + \mathrm{d}_0 x_1)$$

The $\mathrm{d}_1^2 x_0$ term requires evaluation of :

$$\mathrm{d}_1^2 A = \frac{1}{2}\mu\,\mathrm{d}_1 A + \frac{3}{2}[i\frac{c}{\omega_1} - (b_0 + b_1)\omega_1^2](2A\overline{A}\,\mathrm{d}_1 A + A^2\,\mathrm{d}_1 \overline{A})$$
$$= \frac{1}{4}\mu^2 A - 3\mu(b_0 + b_1)\omega_1^2 A^2 \overline{A}$$
$$-\frac{9}{4}\frac{c^2}{\omega_1^2} - 9i(b_0 + b_1)c\omega_1 + \frac{27}{4}(b_0 + b_1)^2\omega_1^4 A^3 \overline{A}^2$$

➢ elimination of secular terms:

$$
\begin{aligned}
d_2 A = & -i\frac{\mu^2}{8\omega_1}A - \frac{3}{4}\frac{c}{\omega_1^2}\mu A^2\overline{A} \\
& +[-\frac{3}{2}c(b_0+b_1) - i\frac{15}{16}\frac{c^2}{\omega_1^3} + \frac{9}{16}i\omega_1^3(b_0+b_1)^2 \\
& +9ib_0^2\omega_1^5(\frac{1}{2}\frac{1}{\omega_2^2-9\omega_1^2+3i\xi\omega_1} \\
& +\frac{1}{\omega_2^2-\omega_1^2+i\xi\omega_1} - \frac{1}{2}\frac{1}{\omega_2^2-\omega_1^2-i\xi\omega_1})]A^3\overline{A}^2
\end{aligned}
$$

- Reconstitution method and parameter reabsorbing:

$$\dot{A} = \varepsilon \, \mathrm{d}_1 \, A + \varepsilon^2 \, \mathrm{d}_2 \, A$$

This equation is multiplied by $\varepsilon^{1/2}$ and quantities transformed back as $\varepsilon^{1/2} A \to A, \varepsilon\mu \to \mu$, thus obtaining a *complex bifurcation equation*:

$$\dot{A} = (\frac{1}{2}\mu - i\frac{\mu^2}{8\omega_1})A + \frac{3}{2}[i\frac{c}{\omega_1} - \frac{3}{4}\frac{c}{\omega_1^2}\mu - (b_0 + b_1)\omega_1^2]A^2\bar{A}$$
$$+[-\frac{3}{2}c(b_0 + b_1) - i\frac{15}{16}\frac{c^2}{\omega_1^3} + \frac{9}{16}i\omega_1^3(b_0 + b_1)^2$$
$$+9ib_0^2\omega_1^5(\frac{1}{2}\frac{1}{\omega_2^2 - 9\omega_1^2 + 3i\xi\omega_1} + \frac{1}{\omega_2^2 - \omega_1^2 + i\xi\omega_1} - \frac{1}{2}\frac{1}{\omega_2^2 - \omega_1^2 - i\xi\omega_1})]A^3\bar{A}^2$$

Using the polar form:

$$A(t) := \frac{1}{2} a(t) \mathrm{e}^{i\theta(t)}$$

and separating the real and imaginary parts, two real *bifurcation equations* follow.

- Amplitude equation:

$$\dot{a} = \frac{1}{2}\mu a - [\frac{3}{8}(b_0 + b_1)\omega_1^2 - \frac{3}{16}\frac{c}{\omega_1^2}\mu]a^3$$
$$+[-\frac{3}{32}c(b_0 + b_1) + \frac{27}{32}b_0^2\xi\omega_1^6(\frac{1}{(\omega_2^2 - 9\omega_1^2)^2 + 9\xi^2\omega_1^2} + \frac{1}{(\omega_2^2 - \omega_1^2)^2 + \xi^2\omega_1^2})]a^5$$

- phase-equation:

$$
\begin{aligned}
a\dot{\vartheta} = & -\frac{1}{8}\frac{\mu^2}{\omega_1}a + \frac{3}{8}\frac{c}{\omega_1}a^3 \\
& +[\frac{9}{256}(b_0+b_1)^2\omega_1^3 - \frac{15}{256}\frac{c^2}{\omega_1^3} \\
& -\frac{9}{32}b_0^2\omega_1^7(\frac{9}{(\omega_2^2-9\omega_1^2)^2+9\xi^2\omega_1^2}+\frac{1}{(\omega_2^2-\omega_1^2)^2+\xi^2\omega_1^2}) \\
& +\frac{9}{32}b_0^2\omega_1^5\omega_2^2(\frac{1}{(\omega_2^2-9\omega_1^2)^2+9\xi^2\omega_1^2}+\frac{1}{(\omega_2^2-\omega_1^2)^2+\xi^2\omega_1^2})]a^5
\end{aligned}
$$

❑ **Note:** The essential dynamics of the original system is governed by a one-dimensional amplitude-equation.

- Response of the system:

$$x = a(t)\cos(\Phi(t)) + a^3(t)[\frac{1}{32}\frac{c}{\omega_1^2}\cos(3\Phi(t)) + \frac{1}{32}(b_0 + b_1)\omega_1 \sin(3\Phi(t))] + \cdots$$

$$y = \frac{3}{4}a^3(t)[\frac{b_0\xi\omega_1^4}{(\omega_2^2-\omega_1^2)^2+\xi^2\omega_1^2}\cos(\Phi(t)) - \frac{b_0\xi\omega_1^4}{(\omega_2^2-9\omega_1^2)^2+9\xi^2\omega_1^2}\cos(3\Phi(t))$$
$$+\frac{b_0\omega_1^3(\omega_1^2-\omega_2^2)}{(\omega_2^2-\omega_1^2)^2+\xi^2\omega_1^2}\sin(\Phi(t)) - \frac{b_0\omega_1^3(9\omega_1^2-\omega_2^2)}{(\omega_2^2-9\omega_1^2)^2+9\xi^2\omega_1^2}\sin(3\Phi(t))] + \cdots$$

where:

$$\Phi(t) := \omega_1 t + \theta(t) .$$

■ Steady solutions:

$$a = a_s = \text{const}\,, \qquad \dot{\theta} =: \kappa = \text{const}$$

are limit cycles, of amplitude a_s and (nonlinear) frequency $\dot{\Phi} = \omega_1 + \kappa = \text{const}$

- Remarks

(a) At leading order, only the active coordinate x, contributes to the motion.
(b) At a higher-order, also the passive coordinate y is triggered. This *does not contribute with its own free evolution,* but rather *is forced by the active x-coordinate*
(c) Taking into account passive coordinates, *does not increase* the dimension of the amplitude equations, but just gives a more accurate description of the dynamics

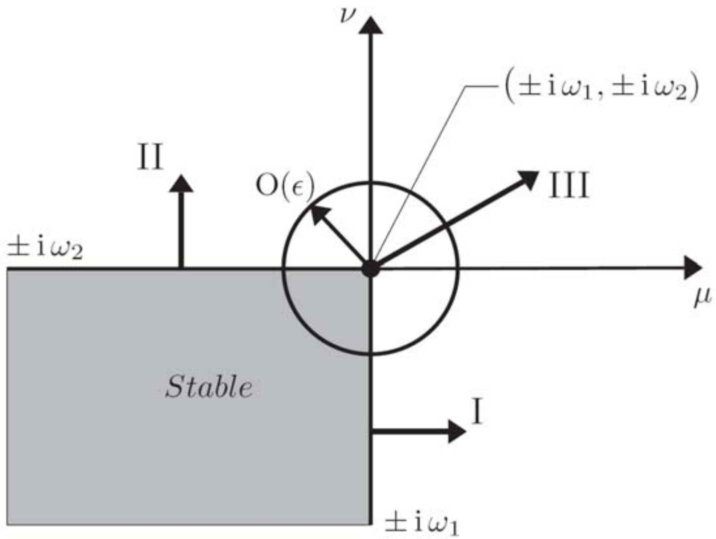
NON-RESONANT DOUBLE-HOPF BIFURCATION

EXAMPLE: TWO COUPLED RAYLEIGH-DUFFING OSCILLATORS, BOTH UNSTABLE

$$\begin{cases} \ddot{x} - \mu \dot{x} + \omega_1^2 x + b_1 \dot{x}^3 + c x^3 - b_0 (\dot{y} - \dot{x})^3 = 0 \\ \ddot{y} - \nu \dot{y} + \omega_2^2 y + b_2 \dot{y}^3 + c y^3 + b_0 (\dot{y} - \dot{x})^3 = 0 \end{cases}$$

whre $\omega_2 \neq r\omega_1, \forall r \in \mathbb{Q}$.

- Linear stability diagram:



- Rescaling:

$$(\mu,\nu)\to(\varepsilon\mu,\varepsilon\nu)\,,\quad (x,y)\to(\varepsilon^{1/2}x,\varepsilon^{1/2}y)$$

- Series expansions:

$$\begin{pmatrix} x(t;\varepsilon) \\ y(t;\varepsilon) \end{pmatrix} = \begin{pmatrix} x_0(t_0,t_1,t_2,\cdots) \\ y_0(t_0,t_1,t_2,\cdots) \end{pmatrix} + \varepsilon \begin{pmatrix} x_1(t_0,t_1,t_2,\cdots) \\ y_1(t_0,t_1,t_2,\cdots) \end{pmatrix} + \varepsilon^2 \begin{pmatrix} x_2(t_0,t_1,t_2,\cdots) \\ y_2(t_0,t_1,t_2,\cdots) \end{pmatrix} + \cdots$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \mathrm{d}_0 + \varepsilon\,\mathrm{d}_1 + \varepsilon^2\,\mathrm{d}_2 + \cdots,\quad \frac{\mathrm{d}^2}{\mathrm{d}t^2} = \mathrm{d}_0^2 + 2\varepsilon\,\mathrm{d}_0\,\mathrm{d}_1 + \varepsilon^2\left(\mathrm{d}_1^2 + 2\,\mathrm{d}_0\,\mathrm{d}_2\right) + \cdots$$

where $t_k := \varepsilon^k t_k$ and $\mathrm{d}_k := \partial/\partial t_k$.

- Perturbation equations:

$$\varepsilon^0:\begin{cases}\mathrm{d}_0^2\,x_0+\omega_1^2x_0=0\\ \mathrm{d}_0^2\,y_0+\omega_2^2y_0=0\end{cases}$$

$$\varepsilon^1:\begin{cases}\mathrm{d}_0^2\,x_1+\omega_1^2x_1=-2\,\mathrm{d}_0\mathrm{d}_1x_0+\mu\,\mathrm{d}_0x_0-b_1(\mathrm{d}_0x_0)^3-cx_0^3+b_0(\mathrm{d}_0y_0-\mathrm{d}_0x_0)^3\\ \mathrm{d}_0^2\,y_1+\omega_2^2y_1=-2\,\mathrm{d}_0\mathrm{d}_1y_0+\nu\,\mathrm{d}_0y_0-b_2(\mathrm{d}_0y_0)^3-cy_0^3-b_0(\mathrm{d}_0y_0-\mathrm{d}_0x_0)^3\end{cases}$$

………

- Generating solution:

$$\begin{cases}x_0=A_1\left(t_1,t_2,\ldots\right)\mathrm{e}^{i\omega_1t_0}+c.c.\\ y_0=A_2\left(t_1,t_2,\ldots\right)\mathrm{e}^{i\omega_2t_0}+c.c.\end{cases}$$

- ε-order:

  - equations:

    Cubic terms produce harmonics $(\omega_1, \omega_2; 3\omega_1, 3\omega_2, \omega_2 \pm 2\omega_1, 2\omega_2 \pm \omega_1)$; among them, only ω_1 in the first equation and ω_2 in the second equation are resonant:

$$\begin{cases} \mathrm{d}_0^2\, x_1 + \omega_1^2 x_1 = f_{1,1}\, \mathrm{e}^{i\omega_1 t_0} + NRT + c.c. \\ \mathrm{d}_0^2\, y_1 + \omega_2^2 y_1 = f_{2,2}\, \mathrm{e}^{i\omega_2 t_0} + NRT + c.c. \end{cases}$$

    where:

$$f_{1,1} := -2i\omega_1\, \mathrm{d}_1 A_1 + i\omega_1 \mu A_1 - 3[c + i(b_0 + b_1)\omega_1^3] A_1^2 \overline{A}_1 - 6b_0 \omega_1 \omega_2^2 A_1 A_2 \overline{A}_2$$

$$f_{2,2} := -2i\omega_2\, \mathrm{d}_1 A_2 + i\nu\omega_2 A_2 - 3[c + i(b_0 + b_2)\omega_2^3] A_2^2 \overline{A}_2 - 6b_0 \omega_1^2 \omega_2 A_1 \overline{A}_1 A_2$$

➢ Zeroing the secular terms requires $f_{1,1} = f_{2,2} = 0$, from which:

$$\begin{cases} \mathrm{d}_1 A_1 = \frac{1}{2}\mu A_1 + \frac{3}{2}[i\frac{c}{\omega_1} - (b_1 + b_0)\omega_1^2]A_1^2\overline{A}_1 - 3b_0\omega_2^2 A_1 A_2 \overline{A}_2 \\ \mathrm{d}_1 A_2 = \frac{1}{2}\nu A_2 + \frac{3}{2}[i\frac{c}{\omega_2} - (b_2 + b_0)\omega_2^2]A_2^2\overline{A}_2 - 3b_0\omega_1^2 A_1 A_2 \overline{A}_1 \end{cases}$$

- First-order solution:

  The previous equations are multiplied by $\varepsilon^{3/2}$ and use is made of the inverse transformations $\varepsilon^{1/2} A_k \to A_k$, $\varepsilon(\mu,\nu) \to (\mu,\nu)$, $\varepsilon\,\mathrm{d}_1 \to \mathrm{D}$, so that $\mathrm{d}_1 A_k \equiv \dot{A}_k$. By using the polar forms:

$$A_k(t) := \frac{1}{2} a_k(t)\,\mathrm{e}^{i\theta_k(t)} \quad k = 1,2$$

  four real bifurcation equations follow.

- Amplitude modulation equations:

$$\begin{cases} \dot{a}_1 = \frac{1}{2}\mu a_1 - \frac{3}{8}(b_0 + b_1)\omega_1^2 a_1^3 - \frac{3}{4} b_0 \omega_2^2 a_1 a_2^2 \\ \dot{a}_2 = \frac{1}{2}\nu a_2 - \frac{3}{8}(b_0 + b_2)\omega_2^2 a_2^3 - \frac{3}{4} b_0 \omega_1^2 a_1^2 a_2 \end{cases}$$

- Phase-modulation equations:

$$\begin{cases} a_1 \dot{\theta}_1 = \frac{3}{8} \frac{c}{\omega_1} a_1^3 \\ a_2 \dot{\theta}_2 = \frac{3}{8} \frac{c}{\omega_2} a_2^3 \end{cases}$$

❑ **Note:** i*n the non-resonant case, the real-amplitude equations are uncoupled from the phase-equations* . Therefore, the essential dynamics of the system is governed by the reduced set of two (RAME) equations.

■ Steady solutions, bifurcation chart and bifurcation diagrams

- Steady motions:

Are the fixed points $a_1 = a_{1s} = \text{const}, a_2 = a_{2s} = \text{const}$. Assume $b_1 = b_2 =: b > 0$, $\beta := b_0 / b > 0$. The steady motions are solutions of:

$$\begin{cases} a_1[\frac{1}{2}\frac{\mu}{b} - \frac{3}{8}(1+\beta)\omega_1^2 a_1^2 - \frac{3}{4}\beta\omega_2^2 a_2^2] = 0 \\ a_2[\frac{1}{2}\frac{\nu}{b} - \frac{3}{8}(1+\beta)\omega_2^2 a_2^2 - \frac{3}{4}\beta\omega_1^2 a_1^2] = 0 \end{cases}$$

- Four essentially different solutions, $(s = T, P_1, P_2, Q)$:

$$(T\ ):\quad a_{1T} = 0, a_{2T} = 0, \forall(\mu, \nu)$$

$$(P_1\ ):\quad a_{1P} = \frac{1}{\omega_1}\sqrt{\frac{4\mu}{3b(1+\beta)}},\quad a_{2P} = 0,\quad \forall\,\nu$$

$$(P_2\ ):\quad a_{1P} = 0,\quad a_{2P} = \frac{1}{\omega_2}\sqrt{\frac{4\nu}{3b(1+\beta)}},\quad \forall\,\mu$$

$$(Q\ ):\quad a_{1Q} = \frac{2}{\omega_1}\sqrt{\frac{2\beta\nu-(1+\beta)\mu}{3b(3\beta^2-2\beta-1)}},\quad a_{2Q} = \frac{2}{\omega_2}\sqrt{\frac{2\beta\mu-(1+\beta)\nu}{3b(3\beta^2-2\beta-1)}}$$

- Meaning of the solutions:

(T) is the *trivial* solution, which corresponds to the equilibrium position of the system.

(P_1) is the mono-modal *periodic* a_1-solution:

$$x = a_{1P}\cos(\Omega_1 t + \theta_{10}), \quad y = 0, \quad \Omega_1 := \omega_1 + \frac{3}{8}\frac{c}{\omega_1}a_{1P}^2$$

(P_2) is the mono-modal *periodic* a_2-solution:

$$x = 0, \quad y = a_{2P}\cos(\Omega_2 t + \theta_{20}), \quad \Omega_2 := \omega_2 + \frac{3}{8}\frac{c}{\omega_2}a_{2P}^2$$

(Q) is a bimodal *quasi-periodic* solution:

$$x = a_{1Q}\cos(\Omega_1 t + \theta_{10}), \; y = a_{2Q}\cos(\Omega_2 t + \theta_{20}), \; \Omega_1 := \omega_1 + \frac{3c}{8\omega_1}a_{1Q}^2, \; \Omega_2 := \omega_2 + \frac{3c}{8\omega_2}a_{2Q}^2$$

since ω_1 and ω_2 are incommensurable.

- Existence domains of the solutions

Since the amplitudes are real and positive:

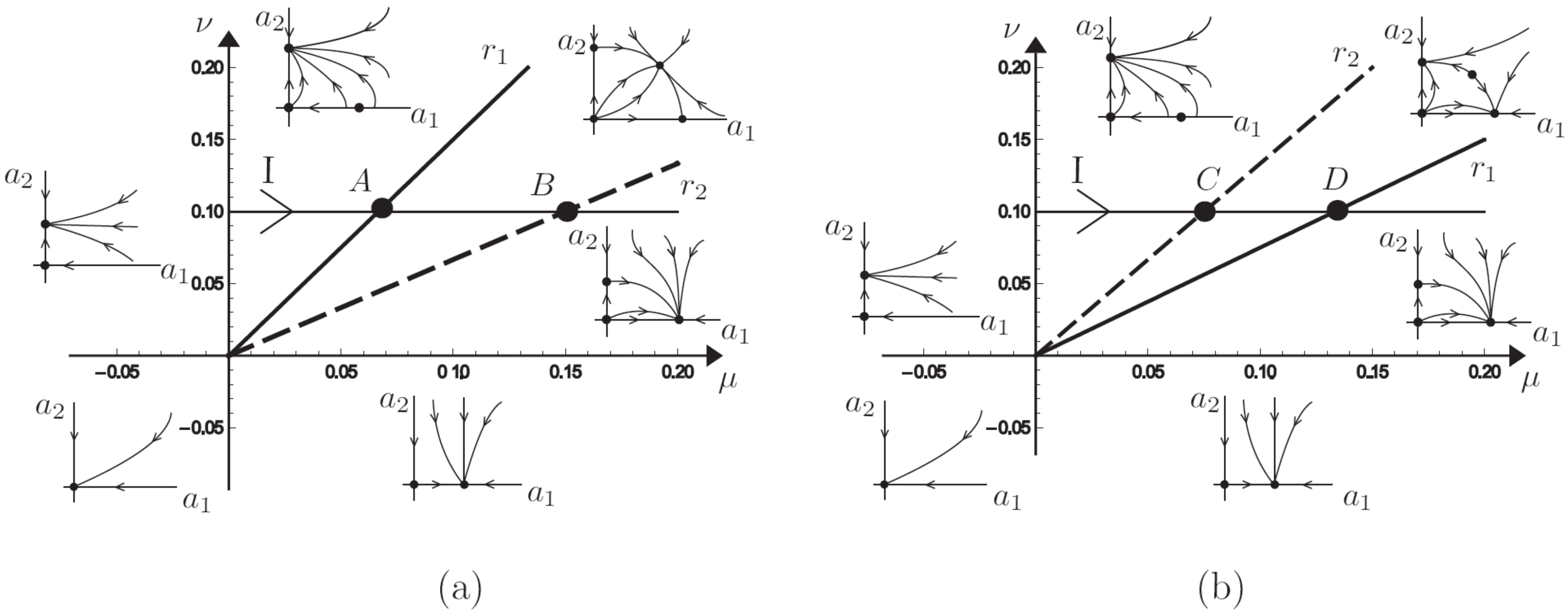
- the T-solution exists in the whole plane
- the P_1-solution is defined in the $\mu \geq 0$ half-plane
- the P_2-solution in the $\nu \geq 0$ half-plane
- the Q-solution requires:

$$\frac{2\beta}{1+\beta}\mu < \nu < \frac{1+\beta}{2\beta}\mu \quad \text{if } \beta < 1$$

$$\frac{1+\beta}{2\beta}\mu < \nu < \frac{2\beta}{1+\beta}\mu \quad \text{if } \beta > 1$$

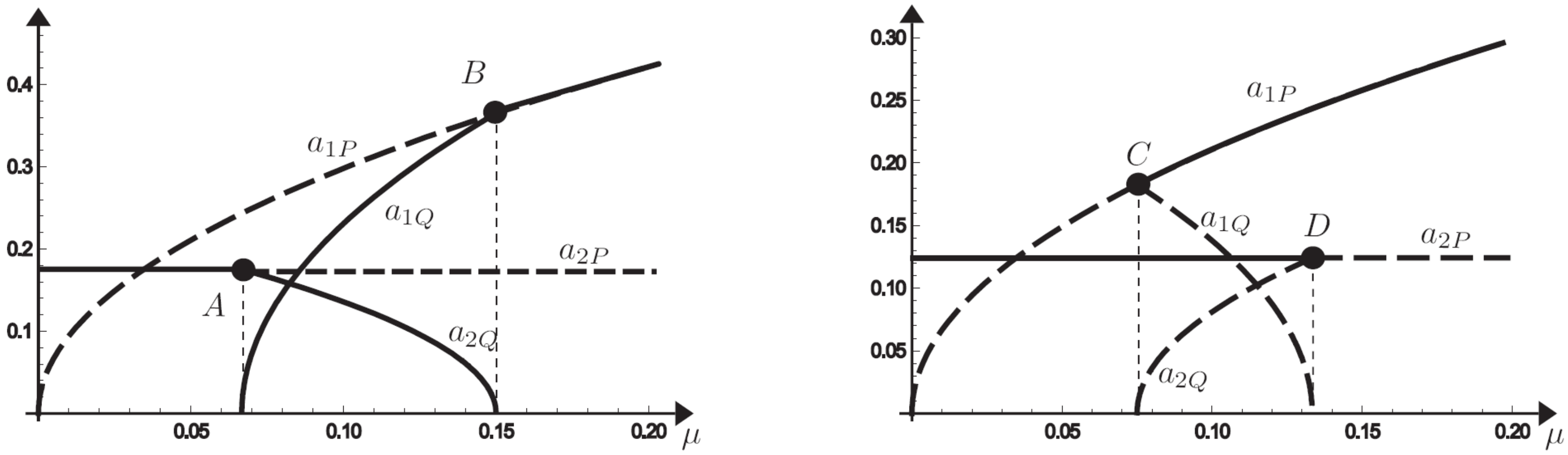
i.e. it exists in the sector bounded by $r_1 := \{(\mu,\nu) \,|\, \nu = (1+\beta)/(2\beta)\mu\}$, $r_2 := \{(\mu,\nu) \,|\, \nu = (2\beta)/(1+\beta)\mu\}$.

- At r_1: $a_{1Q} = 0$, $a_{2Q} = a_{2P}$; at r_2: $a_{1Q} = a_{1P}$, $a_{2Q} = 0$
- r_1 and r_2 are *bifurcation loci*, where a quasi-periodic motion bifurcates from a periodic motion.



Bifurcation chart for: (a) $\beta = 1/2$ and (b) $\beta = 2$

- Planar bifurcation diagrams:



(a) (b)

Bifurcation diagrams for (a) $\beta = 1/2$ and (b) $\beta = 2$; $\nu = 0.1, \omega_1 = 1, \omega_2 = 1.7$; ---- stable, --- unstable

- Stability of steady-solutions

Variation of the amplitude equations:

$$\begin{pmatrix} \delta \dot{a}_1 \\ \delta \dot{a}_2 \end{pmatrix} = \mathbf{J}_s \begin{pmatrix} \delta a_1 \\ \delta a_2 \end{pmatrix}$$

where:

$$\mathbf{J}_s := \begin{pmatrix} \frac{\mu}{2} - \frac{9}{8} b\omega_1^2 (1+\beta) a_{1s}^2 - \frac{3}{4} \beta b \omega_2^2 a_{2s}^2 & -\frac{3}{2} \beta b \omega_2^2 a_{1s} a_{2s} \\ -\frac{3}{2} \beta b \omega_1^2 a_{1s} a_{2s} & \frac{\nu}{2} - \frac{9}{8} b\omega_2^2 (1+\beta) a_{2s}^2 - \frac{3}{4} \beta b \omega_1^2 a_{1s}^2 \end{pmatrix}$$

is the Jacobian evaluated at the steady-solution *s*.
In order that *s* is (asymptotically) stable, both the eigenvalues of $\mathbf{J}_s$ must have negative real part.

For each solution:

- Trivial solution (s=T): $\mathbf{J}_T = \mathrm{diag}[\mu/2, \nu/2]$, i.e. the trivial solution is stable in the third quadrant and unstable elsewhere;

- Periodic solutions (s=P_1,P_2):

$$\mathbf{J}_{P_1} = \mathrm{diag}[-\mu, \frac{\nu}{2} - \frac{\beta\mu}{1+\beta}], \quad \mathbf{J}_{P_2} = \mathrm{diag}[-\nu, \frac{\mu}{2} - \frac{\beta\nu}{1+\beta}]$$

An eigenvalue is always negative; the other vanishes at the straight lines r_2 and r_1. The P_1-solution is stable *below* r_2, and the P_2-solution is stable *above* r_1 .

(continue)

➢ Quasi-periodic solution (*s*=*Q*):

$$\mathrm{tr}[\mathbf{J}_Q] = -\frac{(1+\beta)(\mu+\nu)}{1+3\beta}, \quad \det[\mathbf{J}_Q] = \frac{2\beta\nu-(1+\beta)\mu}{\beta-1}\frac{2\beta\mu-(1+\beta)\nu}{1+3\beta}$$

- ✓ For asymptotic stability $\mathrm{tr}[\mathbf{J}_Q] < 0, \det[\mathbf{J}_Q] > 0$ simultaneously.
- ✓ $\mathrm{tr}[\mathbf{J}_Q] < 0$ in any points of the existence domain;
- ✓ $\det[\mathbf{J}_Q] = 0$ at r_1 and r_2; inside the domain:

  $\det[\mathbf{J}_Q] > 0$ when $\beta < 1$ (*Q*-solution stable),

  $\det[\mathbf{J}_Q] < 0$ when $\beta > 1$ (*Q*-solution unstable).

DIVERGENCE-HOPF BIFURCATION

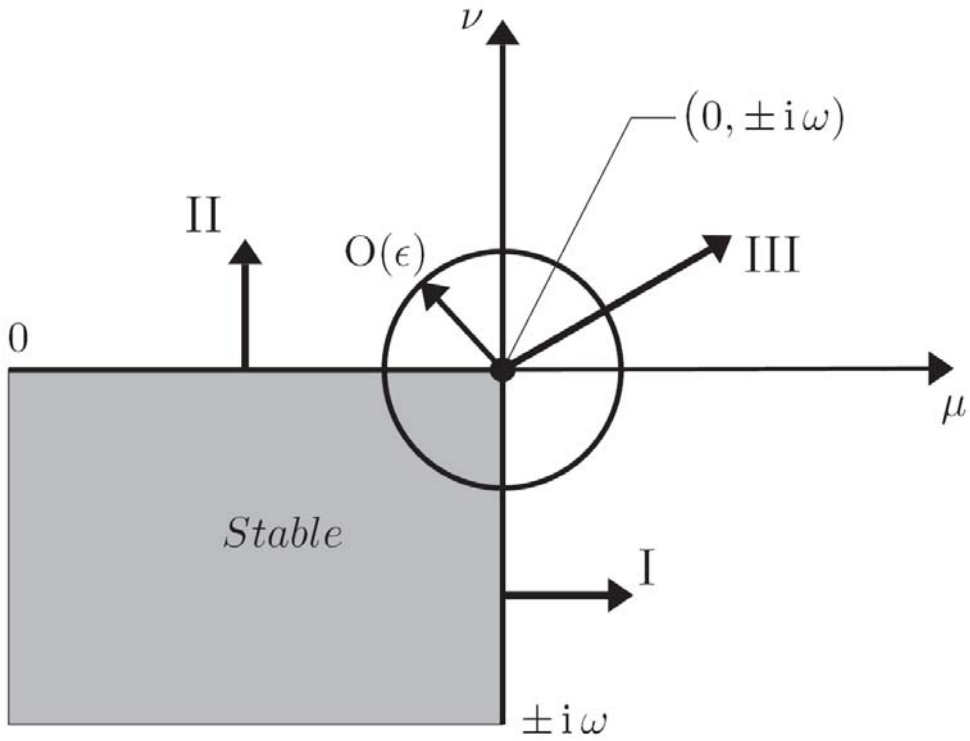
■ EXAMPLE: TWO COUPLED RAYLEIGH-DUFFING OSCILLATORS, BOTH UNSTABLE

The *x*-oscillator undergoes a *dynamic bifurcation*, governed by the parameter μ; the *y*-oscillator, suffers a *static bifurcation*, governed by the parameter ν:

$$\begin{cases} \ddot{x} - \mu\dot{x} + \omega^2 x + b_1\dot{x}^3 + cx^3 - b_0(y-x)^2(\dot{y}-\dot{x}) - c_0(y-x)^3 = 0 \\ \ddot{y} + \xi\dot{y} - \nu y + b_2\dot{y}^3 + cy^3 + b_0(y-x)^2(\dot{y}-\dot{x}) + c_0(y-x)^3 = 0 \end{cases}$$

where $\xi = \mathrm{O}(1) > 0$.

- Discussion on stability:



- Rescaling:

$$(\mu, \nu) \to (\varepsilon\mu, \varepsilon\nu), \quad (x, y) \to (\varepsilon^{1/2}x, \varepsilon^{1/2}y)$$

- Series expansions:

$$\begin{pmatrix} x(t;\varepsilon) \\ y(t;\varepsilon) \end{pmatrix} = \begin{pmatrix} x_0(t_0, t_1, t_2, \cdots) \\ y_0(t_0, t_1, t_2, \cdots) \end{pmatrix} + \varepsilon \begin{pmatrix} x_1(t_0, t_1, t_2, \cdots) \\ y_1(t_0, t_1, t_2, \cdots) \end{pmatrix} + \varepsilon^2 \begin{pmatrix} x_2(t_0, t_1, t_2, \cdots) \\ y_2(t_0, t_1, t_2, \cdots) \end{pmatrix} + \cdots$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \mathrm{d}_0 + \varepsilon\,\mathrm{d}_1 + \varepsilon^2\,\mathrm{d}_2 + \cdots, \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2} = \mathrm{d}_0^2 + 2\varepsilon\,\mathrm{d}_0\,\mathrm{d}_1 + \varepsilon^2\left(\mathrm{d}_1^2 + 2\,\mathrm{d}_0\,\mathrm{d}_2\right) + \cdots$$

where $t_k := \varepsilon^k t_k$ and $\mathrm{d}_k := \partial / \partial t_k$.

- Perturbation equations:

$$\varepsilon^0:\begin{cases}\mathrm{d}_0^2\,x_0+\omega^2x_0=0\\ \mathrm{d}_0^2\,y_0+\xi\,\mathrm{d}_0y_0=0\end{cases}$$

$$\varepsilon^1:\begin{cases}\mathrm{d}_0^2\,x_1+\omega^2x_1=-2\,\mathrm{d}_0\mathrm{d}_1x_0+\mu\,\mathrm{d}_0x_0-b_1(\mathrm{d}_0x_0)^3-cx_0^3\\ \qquad +b_0(y_0-x_0)(\mathrm{d}_0y_0-\mathrm{d}_0x_0)^2+c_0(y_0-x_0)^3\\ \mathrm{d}_0^2\,y_1+\xi\,\mathrm{d}_0y_1=-2\,\mathrm{d}_0\mathrm{d}_1y_0-\xi\,\mathrm{d}_1y_0+\nu y_0-b_2(\mathrm{d}_0y_0)^3-cy_0^3\\ \qquad -b_0(y_0-x_0)(\mathrm{d}_0y_0-\mathrm{d}_0x_0)^2-c_0(y_0-x_0)^3\end{cases}$$

- Generating solution:

$$\begin{cases}x_0=A_1\left(t_1,t_2,\ldots\right)\mathrm{e}^{i\omega t_0}+c.c.\\ y_0=a_2\left(t_1,t_2,\ldots\right)\end{cases}$$

where $A_1\in\mathbb{C}, a_2\in\mathbb{R}$. The x-oscillator experiences a harmonic motion, slowly modulated; the y-oscillator rests in a non-trivial equilibrium position, also modulated.

- ε -order:
  - equations:

$$\begin{cases} \mathrm{d}_0^2 x_1 + \omega^2 x_1 = f_{1,1}\, \mathrm{e}^{i\omega t_0} + NRT + c.c. \\ \mathrm{d}_0^2 y_1 + \xi\, \mathrm{d}_0\, y_1 = f_{2,0} + (NRT + c.c.) \end{cases}$$

where the resonant excitations terms are:

$$f_{1,1} := -2i\omega\, \mathrm{d}_1 A_1 + i\omega\mu A_1$$
$$-[3(c+c_0) + ib_0\omega + 3ib_1\omega^3] A_1^2 \overline{A}_1 - (3c_0 + ib_0\omega) A_1 a_2^2$$
$$f_{2,0} := -\xi\, \mathrm{d}_1 a_2 + \nu A_2 - (c+c_0) a_2^3 - 6c_0 A_1 \overline{A}_1 a_2$$

➢ Removing secular terms requires: $f_{1,1}=0$ and $f_{2,0}=0$, from which:

$$\mathrm{d}_1 A_1 = \frac{1}{2}\mu A_1 + \frac{1}{2}\left[\frac{3}{\omega} i(c+c_0) - b_0 - 3b_1\omega^2\right] A_1^2 \bar{A}_1 - (3c_0 + i b_0 \omega) A_1 a_2^2$$

$$\mathrm{d}_1 a_2 = \frac{1}{\xi}\left[\nu A_2 - (c+c_0) a_2^3 - 6c_0 A_1 \bar{A}_1 a_2\right]$$

- Parameter reabsorbing:

  By multiplying the equations by $\varepsilon^{3/2}$ and using $\varepsilon^{1/2} A_1 \to A_1, \varepsilon^{1/2} a_2 \to a_2$, together with $\varepsilon(\mu,\nu) \to (\mu,\nu), \varepsilon\,\mathrm{d}_1 \to \mathrm{D}$, and by expressing A_1 in the polar form, three real bifurcation equations follows.

➢ Two amplitude-equations:

$$\begin{cases} \dot{a}_1 = \frac{1}{2}\mu a_1 - \frac{1}{8}(b_0 + 3b_1\omega_1^2)a_1^3 - \frac{1}{2}b_0 a_1 a_2^2 \\ \dot{a}_2 = \frac{1}{\xi}\nu a_2 - \frac{1}{\xi}(c + c_0)\omega_2^2 a_2^3 - \frac{3}{2\xi}c_0 a_1^2 a_2 \end{cases}$$

➢ One phase equation:

$$a_1\dot{\theta}_1 = \frac{3}{8}\frac{c + c_0}{\omega}a_1^3 + \frac{3}{2}\frac{c_0}{\omega}a_1 a_2^2$$

- **Note:** The amplitude equations governing the non-resonant double-Hopf bifurcation and the divergence-Hopf bifurcations have the same (normal) form.

EXAMPLE: RAYLEIGH-DUFFING COUPLED OSCILLATORS IN 1:1 OR 1:3 INTERNAL RESONANCE

$$\begin{cases} \ddot{x} - \mu\dot{x} + \omega_1^2 x + b_1\dot{x}^3 + cx^3 - b_0(\dot{y} - \dot{x})^3 = 0 \\ \ddot{y} - \nu\dot{y} + \omega_2^2 y + b_2\dot{y}^3 + cy^3 + b_0(\dot{y} - \dot{x})^3 = 0 \end{cases}$$

where:

$$\omega_2 := \widehat{\omega}_2 + \varepsilon\sigma, \quad \widehat{\omega}_2 := r\omega_1, \quad \sigma = \mathrm{O}(1)$$

in which the *detuning* σ is the *third bifurcation parameter*.

- Perturbation equations:

By following the same steps of the non-resonant case, we get:

$$\varepsilon^0 : \begin{cases} d_0^2 x_0 + \omega_1^2 x_0 = 0 \\ d_0^2 y_0 + \hat{\omega}_2^2 y_0 = 0 \end{cases}$$

$$\varepsilon^1 : \begin{cases} d_0^2 x_1 + \omega_1^2 x_1 = -2\, d_0 d_1 x_0 + \mu\, d_0 x_0 - b_1 (d_0 x_0)^3 - c x_0^3 + b_0 (d_0 y_0 - d_0 x_0)^3 \\ d_0^2 y_1 + \hat{\omega}_2^2 y_1 = -2\, d_0 d_1 y_0 + \nu\, d_0 y_0 - b_2 (d_0 y_0)^3 - c y_0^3 - b_0 (d_0 y_0 - d_0 x_0)^3 \\ \qquad -2 \hat{\omega}_2 \sigma y_0 \end{cases}$$

- Generating solution:

$$\begin{cases} x_0 = A_1 \left(t_1, t_2, \ldots \right) e^{i \omega_1 t_0} + c.c. \\ y_0 = A_2 \left(t_1, t_2, \ldots \right) e^{i \hat{\omega}_2 t_0} + c.c. \end{cases}$$

- ε -order:
  - equations:

The harmonics $(\omega_1, \widehat{\omega}_2; 3\omega_1, 3\widehat{\omega}_2, \widehat{\omega}_2 \pm 2\omega_1, 2\widehat{\omega}_2 \pm \omega_1)$ arise:

$$
\begin{cases}
\mathrm{d}_0^2 x_1 + \omega_1^2 x_1 = f_{1,1}\,\mathrm{e}^{i\omega_1 t_0} + f_{1,2}\,\mathrm{e}^{i\widehat{\omega}_2 t_0} + f_{1,30}\,\mathrm{e}^{3i\omega_1 t_0} + f_{1,03}\,\mathrm{e}^{3i\widehat{\omega}_2 t_0} \\
\qquad + f_{1,21}\,\mathrm{e}^{i(2\omega_1 + \widehat{\omega}_2)t_0} + f_{1,\bar{2}1}\,\mathrm{e}^{i(\widehat{\omega}_2 - 2\omega_1)t_0} \\
\qquad + f_{1,12}\,\mathrm{e}^{i(2\widehat{\omega}_2 + \omega_1)t_0} + f_{1,\bar{1}2}\,\mathrm{e}^{i(2\widehat{\omega}_2 - \omega_1)t_0} + c.c. \\
\mathrm{d}_0^2 y_1 + \widehat{\omega}_2^2 y_1 = f_{2,1}\,\mathrm{e}^{i\omega_1 t_0} + f_{2,2}\,\mathrm{e}^{i\widehat{\omega}_2 t_0} + f_{2,30}\,\mathrm{e}^{3i\omega_1 t_0} + f_{2,03}\,\mathrm{e}^{3i\widehat{\omega}_2 t_0} \\
\qquad + f_{2,21}\,\mathrm{e}^{i(2\omega_1 + \widehat{\omega}_2)t_0} + f_{2,\bar{2}1}\,\mathrm{e}^{i(\widehat{\omega}_2 - 2\omega_1)t_0} \\
\qquad + f_{2,12}\,\mathrm{e}^{i(2\widehat{\omega}_2 + \omega_1)t_0} + f_{2,\bar{1}2}\,\mathrm{e}^{i(2\widehat{\omega}_2 - \omega_1)t_0} + c.c.
\end{cases}
$$

where:

$$f_{1,1} := -2i\omega_1 \, \mathrm{d}_1 A_1 + i\omega_1 \mu A_1 - 3[c + i\omega_1^3 (b_0 + b_1)] A_1^2 \overline{A}_1 - 6ib_0 \omega_1 \omega_2^2 A_1 A_2 \overline{A}_2$$

$$f_{2,2} := -2i\omega_2 \, \mathrm{d}_1 A_2 + \omega_2 (i\nu - 2\sigma) A_2 - 3[c + i\omega_2^3 (b_0 + b_2)] A_2^2 \overline{A}_2 - 6ib_0 \omega_1^2 \omega_2 A_1 \overline{A}_1 A_2$$

$$f_{1,2} := 6ib_0 \omega_1^2 \omega_2 A_1 A_2 \overline{A}_1 + 3ib_0 \omega_2^3 A_2^2 \overline{A}_2, \quad f_{2,1} := 3ib_0 \omega_1^3 A_1^2 \overline{A}_1 + 6ib_0 \omega_1 \omega_2^2 A_1 A_2 \overline{A}_2$$

$$f_{1,30} := [-c + i\omega_1^3 (b_0 + b_1)] A_1^3, \quad f_{2,30} := -ib_0 \omega_1^3 A_1^3$$

$$f_{1,03} := -ib_0 \omega_2^3 A_2^3, \quad f_{2,03} := [-c + i\omega_2^3 (b_0 + b_2)] A_2^3$$

$$f_{1,21} = -f_{2,21} := -3ib_0 \omega_1^2 \omega_2 A_1^2 A_2, \quad f_{1,\bar{2}1} = -f_{2,\bar{2}1} := -3ib_0 \omega_1^2 \omega_2 \overline{A}_1^2 A_2$$

$$f_{1,12} = -f_{2,12} := 3ib_0 \omega_1 \omega_2^2 A_1 A_2^2, \quad f_{1,\bar{1}2} = -f_{2,\bar{1}2} := -3ib_0 \omega_1 \omega_2^2 \overline{A}_1 A_2^2$$

➢ Zeroing secular terms:

In a first-order analysis it does not need to compute all the *f*-coefficients, but only the resonant ones. By inspection:

$$\begin{cases} f_{1,1} + \delta_{r1} (f_{1,2} + f_{1,2\bar{1}} + f_{1,\bar{1}2}) + \delta_{r3} f_{1,\bar{2}1} = 0 \\ f_{2,2} + \delta_{r1} (f_{2,1} + f_{2,2\bar{1}} + f_{2,\bar{1}2}) + \delta_{r3} f_{2,30} = 0 \end{cases}$$

where δ_{rk} is the Kronecker symbol ($\delta_{rk} = 1$ if $r = k$, $\delta_{rk} = 0$ if $r \neq k$).

■ **The r=1 case**

The complex AME read:

$$\begin{cases} \mathrm{d}_1 A_1 = \frac{1}{2}\mu A_1 + \frac{3}{2}[i\frac{c}{\omega_1} - (b_1 + b_0)\omega_1^2]A_1^2\overline{A}_1 - 3b_0\omega_1^2 A_1 A_2 \overline{A}_2 \\ \qquad +3b_0\omega_1^2 A_1\overline{A}_1 A_2 + \frac{3}{2}b_0\omega_1^2 A_1^2\overline{A}_2 - \frac{3}{2}b_0\omega_1^2\overline{A}_1 A_2^2 + \frac{3}{2}b_0\omega_1^2 A_2^2\overline{A}_2 \\ \mathrm{d}_1 A_2 = (\frac{1}{2}\nu + i\sigma)A_2 + \frac{3}{2}[i\frac{c}{\omega_1} - (b_2 + b_0)\omega_1^2]A_2^2\overline{A}_2 - 3b_0\omega_1^2 A_1\overline{A}_1 A_2 \\ \qquad +3b_0\omega_1^2 A_1 A_2\overline{A}_2 + \frac{3}{2}b_0\omega_1^2 A_1^2\overline{A}_1 - \frac{3}{2}b_0\omega_1^2 A_1^2\overline{A}_2 + \frac{3}{2}b_0\omega_1^2\overline{A}_1 A_2^2 \end{cases}$$

in which $\widehat{\omega}_2 = \omega_1$ has been considered. By absorbing the parameter ε, using the polar representation and separating the real and imaginary parts, four real bifurcation equations follow:

$$
\begin{cases}
\dot{a}_1 = \frac{1}{2}\mu a_1 - \frac{3}{8}(b_0 + b_1)\omega_1^2 a_1^3 - \frac{3}{8}b_0\omega_1^2[2+\cos(2\theta_1 - 2\theta_2)]a_1 a_2^2 \\
\quad + \frac{9}{8}b_0\omega_1^2 a_1^2 a_2 \cos(\theta_1 - \theta_2) + \frac{3}{8}b_0\omega_1^2 a_2^3 \cos(\theta_1 - \theta_2) \\
\dot{a}_2 = \frac{1}{2}\nu a_2 - \frac{3}{8}(b_0 + b_2)\omega_1^2 a_2^3 - \frac{3}{8}b_0\omega_1^2[2+\cos(2\theta_1 - 2\theta_2)]a_1^2 a_2 \\
\quad + \frac{3}{8}b_0\omega_1^2 a_1^3 \cos(\theta_1 - \theta_2) + \frac{9}{8}b_0\omega_1^2 a_1 a_2^2 \cos(\theta_1 - \theta_2) \\
a_1\dot{\theta}_1 = \frac{3}{8}\frac{c}{\omega_1}a_1^3 + \frac{3}{8}b_0\omega_1^2 a_1^2 a_2 \sin(\theta_1 - \theta_2) + \frac{3}{8}b_0\omega_1^2 a_2^3 \sin(\theta_1 - \theta_2) \\
\quad - \frac{3}{8}b_0\omega_1^2 a_1 a_2^2 \sin(2\theta_1 - 2\theta_2) \\
a_2\dot{\theta}_2 = \sigma a_2 + \frac{3}{8}\frac{c}{\omega_1}a_2^3 + \frac{3}{8}b_0\omega_1^2 a_1^3 \sin(\theta_1 - \theta_2) + \frac{3}{8}b_0\omega_1^2 a_1 a_2^2 \sin(\theta_1 - \theta_2) \\
\quad - \frac{3}{8}b_0\omega_1^2 a_1^2 a_2 \sin(2\theta_1 - 2\theta_2)
\end{cases}
$$

- **Note:** *the real-amplitude equations are coupled with the phase-equations*
.

Since phases appear as a linear combination, we introduce a *phase-combination*:

$$\gamma := \theta_1 - \theta_2$$

and recombine the phase-equations according $\dot{\gamma} = \dot{\theta}_1 - \dot{\theta}_2$. We obtain:

➢ *three* RAME in the state-variables $\{a_1, a_2, \gamma\}$:

$$\begin{cases} \dot{a}_1 = \frac{1}{2}\mu a_1 - \frac{3}{8}(b_0 + b_1)\omega_1^2 a_1^3 - \frac{3}{8} b_0 \omega_1^2 [2 + \cos 2\gamma] a_1 a_2^2 \\ \quad + \frac{9}{8} b_0 \omega_1^2 a_1^2 a_2 \cos\gamma + \frac{3}{8} b_0 \omega_1^2 a_2^3 \cos\gamma \\ \dot{a}_2 = \frac{1}{2}\nu a_2 - \frac{3}{8}(b_0 + b_2)\omega_1^2 a_2^3 - \frac{3}{8} b_0 \omega_1^2 [2 + \cos 2\gamma] a_1^2 a_2 \\ \quad + \frac{3}{8} b_0 \omega_1^2 a_1^3 \cos\gamma + \frac{9}{8} b_0 \omega_1^2 a_1 a_2^2 \cos\gamma \\ a_1 a_2 \dot{\gamma} = -\sigma a_1 a_2 + \frac{3}{8}(\frac{c}{\omega_1} + b_0 \omega_1^2 \sin 2\gamma) a_1^3 a_2 + \frac{3}{8}(b_0 \omega_1^2 \sin 2\gamma - \frac{c}{\omega_1}) a_1 a_2^3 \\ \quad - \frac{3}{8} b_0 \omega_1^2 a_1^4 \sin\gamma - \frac{3}{4} b_0 \omega_1^2 a_1^2 a_2^2 \sin\gamma - \frac{3}{8} b_0 \omega_1^2 a_2^4 \sin\gamma \end{cases}$$

➢two phase-equations:

$$\begin{cases} a_1\dot{\theta}_1 = \frac{3}{8}\frac{c}{\omega_1}a_1^3 + \frac{3}{8}b_0\omega_1^2 a_1^2 a_2 \sin\gamma + \frac{3}{8}b_0\omega_1^2 a_2^3 \sin\gamma - \frac{3}{8}b_0\omega_1^2 a_1 a_2^2 \sin 2\gamma \\ a_2\dot{\theta}_2 = \sigma a_2 + \frac{3}{8}\frac{c}{\omega_1}a_2^3 + \frac{3}{8}b_0\omega_1^2 a_1^3 \sin\gamma + \frac{3}{8}b_0\omega_1^2 a_1 a_2^2 \sin\gamma - \frac{3}{8}b_0\omega_1^2 a_1^2 a_2 \sin 2\gamma \end{cases}$$

Once the RAME have been solved, the phase-equations can be integrated by quadrature.

❑ **Note:** while the RAME of a *non-resonant* system are pure-amplitude equations, those of a *resonant* system are mixed-amplitude-phase equations.

■ **The r=3 case**

In a similar way, the complex AME are found to be:

$$\begin{cases} \mathrm{d}_1 A_1 = \frac{1}{2}\mu A_1 + \frac{3}{2}[i\frac{c}{\omega_1} - (b_1 + b_0)\omega_1^2]A_1^2\bar{A}_1 - 27b_0\omega_1^2 A_1 A_2 \bar{A}_2 - \frac{9}{2}b_0\omega_1^2 \bar{A}_1^2 A_2 \\ \mathrm{d}_1 A_2 = (\frac{1}{2}\nu + i\sigma)A_2 + \frac{1}{2}[i\frac{c}{\omega_1} - 27(b_2 + b_0)\omega_1^2]A_2^2\bar{A}_2 - 3b_0\omega_1^2 A_1 A_2 \bar{A}_1 - \frac{1}{6}b_0\omega_1^2 A_1^3 \end{cases}$$

in which $\hat{\omega}_2 = 3\omega_1$ has been substituted.

After parameter reabsorbing, and use of the polar representation, we obtain four real bifurcation equations:

$$
\begin{cases}
\dot{a}_1 = \dfrac{1}{2}\mu a_1 - \dfrac{3}{8}(b_0 + b_1)\omega_1^2 a_1^3 - \dfrac{27}{4} b_0 \omega_1^2 a_1 a_2^2 - \dfrac{9}{8} b_0 \omega_1^2 a_1^2 a_2 \cos(3\theta_1 - \theta_2) \\
\dot{a}_2 = \dfrac{1}{2}\nu a_2 - \dfrac{27}{8}(b_0 + b_2)\omega_1^2 a_2^3 - \dfrac{3}{4} b_0 \omega_1^2 a_1^2 a_2 - \dfrac{1}{24} b_0 \omega_1^2 a_1^3 \cos(3\theta_1 - \theta_2) \\
a_1\dot{\theta}_1 = \dfrac{3}{8}\dfrac{c}{\omega_1} a_1^3 + \dfrac{9}{8} b_0 \omega_1^2 a_1^2 a_2 \sin(3\theta_1 - \theta_2)] \\
a_2\dot{\theta}_2 = \sigma a_2 + \dfrac{1}{8}\dfrac{c}{\omega_1} a_2^3 - \dfrac{1}{24} b_0 \omega_1^2 a_1^3 \sin(3\theta_1 - \theta_2)]
\end{cases}
$$

They suggest the following definition for the phase-combination:

$$\gamma := 3\theta_1 - \theta_2$$

➢ The RAME are:

$$\begin{cases} \dot{a}_1 = \frac{1}{2}\mu a_1 - \frac{3}{8}(b_0 + b_1)\omega_1^2 a_1^3 - \frac{27}{4}b_0\omega_1^2 a_1 a_2^2 - \frac{9}{8}b_0\omega_1^2 a_1^2 a_2 \cos\gamma \\ \dot{a}_2 = \frac{1}{2}\nu a_2 - \frac{27}{8}(b_0 + b_2)\omega_1^2 a_2^3 - \frac{3}{4}b_0\omega_1^2 a_1^2 a_2 + \frac{1}{24}(b_0 + b_1)\omega_1^2 a_1^3 \cos\gamma \\ a_1 a_2\dot{\gamma} = -\sigma a_1 a_2 + \frac{9}{8}\frac{c}{\omega_1}a_1^3 a_2 - \frac{1}{8}\frac{c}{\omega_1}a_1 a_2^3 + \frac{1}{24}b_0\omega_1^2 a_1^4 \sin\gamma + \frac{27}{8}b_0\omega_1^2 a_1^2 a_2^2 \sin\gamma \end{cases}$$

➢ The phase-equations are:

$$\begin{cases} a_1\dot{\theta}_1 = \frac{3}{8}\frac{c}{\omega_1}a_1^3 + \frac{9}{8}b_0\omega_1^2 a_1^2 a_2 \sin\gamma \\ a_2\dot{\theta}_2 = \sigma a_2 + \frac{1}{8}\frac{c}{\omega_1}a_2^3 - \frac{1}{24}b_0\omega_1^2 a_1^3 \sin\gamma \end{cases}$$

- Response (r =1,3 cases)

  After integration, the RAME furnish $a_1(t), a_2(t), \gamma(t)$; successively, the phase equations give $\theta_1(t), \theta_2(t)$. The response read:

$$\begin{cases} x = a_1(t)\cos(\Phi_1(t)) + h.o.t. \\ y = a_2(t)\cos(\Phi_2(t)) + h.o.t. \end{cases}$$

where:

$$\Phi_1(t) := \omega_1 t + \theta_1(t), \quad \Phi_2(t) := \hat{\omega}_2 t + \theta_2(t)$$

are total phases.

■ Steady-state solutions and fixed points of RAME

- The RAME, are of the following type:

$$\begin{cases} \dot{a}_1 = F_1(a_1, a_2, \gamma) \\ \dot{a}_2 = F_2(a_1, a_2, \gamma) \\ a_1 a_2 \dot{\gamma} = G(a_1, a_2, \gamma) \end{cases}$$

and phase-equations are of the type:

$$\begin{cases} a_1 \dot{\theta}_1 = H_1(a_1, a_2, \gamma) \\ a_2 \dot{\theta}_2 = H_2(a_1, a_2, \gamma) \end{cases}$$

❑ **Note:** The RAME can be put in the standard form $\dot{\mathbf{z}} = \mathbf{F}(\mathbf{z})$, with $\mathbf{z} := (a_1, a_2, \gamma)$, if and only if $a_1 \neq 0,\ a_2 \neq 0$ (complete solutions).

❑ **Note:** in incomplete solutions ($a_1 = 0$, and/or $a_2 = 0$), the phases of the zero-amplitudes remains undetermined; however, they are inessential.

- The fixed points $(a_{1s}, a_{2s}, \gamma_s) = \text{const}$ of RAME are solutions of:

$$\begin{cases} F_1(a_{1s}, a_{2s}, \gamma_s) = 0 \\ F_2(a_{1s}, a_{2s}, \gamma_s) = 0 \\ G(a_{1s}, a_{2s}, \gamma_s) = 0 \end{cases}$$

Consequently, the associated phases (if determined) are *linearly varying* functions:

$$\theta_{1s}(t) = \kappa_{1s} t + \theta_{1s}^0, \quad \theta_{2s}(t) = \kappa_{2s} t + \theta_{2s}^0$$

with $(\kappa_{1s}, \kappa_{2s}) = \text{const}$ the *frequency corrections.*

- For a complete solution, we prove that *the* (non-trivial) *fixed points of the RAME are periodic motions for the original system* (for incomplete solution, this is a trivial matter). Indeed:
  - a constant phase-difference:

$$\gamma_s := r\theta_{1s} - \theta_{2s} = r(\kappa_{1s} t + \theta_{1s}^0) - (\kappa_{2s} t + \theta_{2s}^0) = \text{const} \qquad r = 1,3$$

    entails a relation between frequency corrections and initial phases:

$$r\kappa_{1s} - \kappa_{2s} = 0, \quad r\theta_{1s}^0 - \theta_{2s}^0 = \gamma_s$$

  - consequently, since $\hat{\omega}_2 = r\omega_1$, the total phases read:

$$\Phi_1(t) := \omega_1 t + \theta_{1s}(t) = (\omega_1 + \kappa_{1s})t + \theta_{1s}^0$$

$$\Phi_2(t) := \hat{\omega}_2 t + \theta_{2s}(t) = (\hat{\omega}_2 + \kappa_{2s})t + \theta_{2s}^0 = [r(\omega_1 + \kappa_{1s})t + \theta_{2s}^0]$$

    i.e. *the nonlinear frequencies* Ω_k *are in the same integer ratio r as the linear frequencies* ω_k*:*

$$\Omega_{1s} := \omega_1 + \kappa_{1s}, \quad \Omega_2 := \hat{\omega}_2 + \kappa_{2s} = r\Omega_{1s}$$

The steady response, therefore, is periodic, and it reads:

$$\begin{cases} x = a_1(t)\cos(\Omega_{1s}t + \theta_{1s}^0) + h.o.t. \\ y = a_2(t)\cos[r(\Omega_{1s}t + \theta_{1s}^0) - \gamma_s] + h.o.t. \end{cases}$$

- **Note:** the phase difference γ_s is given by the solution; however, an initial phase, e.g. θ_{1s}^0 remains undetermined, since the limit cycle can be traveled starting from any of its points.

■ Finding the fixed points of RAME

- In the r=1 case, the RAME admit:

  - (T) the trivial solution:

$$a_{1T} = a_{2T} = 0, \quad \forall \gamma_T, \quad \forall(\mu, \nu, \sigma)$$

with the phase-difference γ being undetermined.

  - (P) a number of *bimodal* (or complete) periodic solutions:

$$a_{1P} = a_{1P}(\mu, \nu, \sigma), \quad a_{2P} = a_{2P}(\mu, \nu, \sigma), \quad \gamma_P = \gamma_P(\mu, \nu, \sigma)$$

with associated, determined phases θ_{1P} and θ_{2P}.

- In the r=3 case the RAME admit:

  - (T) the trivial solution:

$$a_{1T} = a_{2T} = 0, \quad \forall \gamma_T, \quad \forall (\mu, \nu, \sigma)$$

  - (M) a *mono-modal* (incomplete) periodic solution:

$$a_{1M} = 0, \quad a_{2M} = a_{2M}(\nu), \quad \forall \gamma_M$$

    with:

$$\theta_{2M} = \theta_{2M}(\sigma, \nu), \quad \forall \theta_{1M}$$

  - (P) one or more *bimodal* (complete) periodic solutions:

$$a_{1P} = a_{1P}(\mu, \nu, \sigma), \quad a_{2P} = a_{2P}(\mu, \nu, \sigma), \quad \gamma_P = \gamma_P(\mu, \nu, \sigma)$$

    with associated phases θ_{1P} and θ_{2P}.

■ Stability of steady solutions

It needs to distinguish:

- *The steady-solution is complete* (s=P): since all quantities are determined, and the RAME can be put in the normal form $\dot{\mathbf{z}} = \mathbf{F}(\mathbf{z})$, with $\mathbf{z} := (a_1, a_2, \gamma)$, stability is governed by the variational equation:

$$\delta\dot{\mathbf{z}} = \mathbf{J}_P \delta\mathbf{z}$$

A zero eigenvalue of $\mathbf{J}_P$ denotes a branching of a new *periodic* solution; a pair of purely imaginary eigenvalues denotes a branching of a *quasi-periodic* solution (i.e. a periodically modulated periodic motion).

- *The steady-solution is incomplete* (s=T,M): since γ_s is undetermined, and the RAME are *not* in standard form, use of the (not reduced) AME must be made. Examples are given below.

- Stability of the trivial solution (r=1,3 cases)

The variation of the AME, based on $A_{1T} = A_{2T} = 0$, reads:

$$\begin{cases} \delta \dot{A}_1 = \frac{1}{2}\mu \delta A_1 \\ \delta \dot{A}_2 = (\frac{1}{2}\nu + i\sigma)\delta A_2 \end{cases}$$

whose solution is:

$$\delta A_1 = \delta \hat{A}_1 \exp(\frac{1}{2}\mu t), \quad \delta A_2 = \delta \hat{A}_2 \exp[(\frac{1}{2}\nu + i\sigma)t]$$

with $\delta \hat{A}_1, \delta \hat{A}_2$ constants. The trivial solution is therefore stable when $\mu < 0, \nu < 0$.

- Stability of the mono-modal solution (r=3 case)

  - The variation of the AME, based on:

$$A_{1M} = 0, \quad A_{2M} = A_{2M} := \frac{1}{2} a_{2M} \exp[i(\kappa_{2M} t + \theta_{2M}^0)]$$

assumes the following (uncoupled) form:

$$\begin{cases} \delta \dot{A}_1 = (R_1 + R_2 \frac{1}{4} a_{2M}^2) \delta A_1 \\ \delta \dot{A}_2 = (C_1 + C_2 \frac{1}{4} a_{2M}^2) \delta A_2 + C_3 \frac{1}{4} a_{2M}^2 \exp[2i(\kappa_{2M} t + \theta_{2M}^0)] \delta \overline{A}_2 \end{cases}$$

where $R_j \in \mathbb{R}, C_j := R_j + iI_j \in \mathbb{C}$ are coefficients.

- **Note:** due to the presence of the frequency correction κ_{2M}, $A_{2M} \neq \text{const}$; consequently, the second variational equation depends on time.

- *To render the second equation autonomous*, a change of variable is performed:

$$\delta A_2 = \delta B_2 \exp[i(\alpha t + \beta)]$$

with α, β to be determined. By requiring the coefficients are independent of time, it follows: $\alpha = \kappa_{2M}$; moreover $\beta = \theta^0_{2M}$ is taken for simplicity.

- In the new variables, the variational equations read:

$$\begin{cases} \delta \dot{A}_1 = (R_1 + R_2 \frac{1}{4} a_{2M}^2)\delta A_1 \\ \delta \dot{B}_2 = (C_1 + C_2 \frac{1}{4} a_{2M}^2 - i\kappa_{2M})\delta B_2 + C_3 \frac{1}{4} a_{2M}^2 \delta \bar{B}_2 \end{cases}$$

- Since the equations are linear, a Cartesian representation is better suited:

$$\delta A_1 = p_1 + iq_1, \quad \delta B_2 = p_2 + iq_2$$

leading to four real variational equations:

$$\begin{pmatrix} \dot{p}_1 \\ \dot{q}_1 \\ \dot{p}_2 \\ \dot{q}_2 \end{pmatrix} = \begin{pmatrix} J_{11} & 0 & & \\ 0 & J_{22} & & \\ & & J_{33} & J_{34} \\ & & J_{43} & J_{44} \end{pmatrix} \begin{pmatrix} p_1 \\ q_1 \\ p_2 \\ q_2 \end{pmatrix}$$

where:

$$J_{11} = J_{22} := R_1 + \frac{R_2}{4} a_{2M}^2$$

$$J_{33} = R_1 + (R_2 + R_3)\frac{a_{2M}^2}{4}, \quad J_{34} = -I_1 + (-I_2 + I_3)\frac{a_{2M}^2}{4} + \kappa_{2M}$$

$$J_{33} = I_1 + (I_2 + I_3)\frac{a_{2M}^2}{4} - \kappa_{2M}, \quad J_{34} = R_1 + (R_2 - R_3)\frac{a_{2M}^2}{4},$$

The eigenvalues (four real, or two real and two complex conjugate), govern the stability of the *M*-solution.