Chapter 1
The Discrete Fourier Transform

1 Introduction

The spectral analysis of time series is a complementary technique to Box-Jenkins modelling which emphatizes different aspects of the data set. It has many applications in all fields of Science and Technology; for example, it is used to look for cyclical (or periodical) components in time series that are not caused by seasonality, in a unending and (almost always) unsuccessful look for regularity and reduction of uncertainty in life. In these notes, we will focus on the estimation of the spectral density that can help in the analysis of time series.

The main tool of spectral analysis is the discrete Fourier Analysis, from which we will study some essential aspects. A good reference of Fourier Analysis with no much mathematical background is Weaver [8]. First at all, we should point out that there are three very related theories under the name of Fourier:

- **Fourier Series expansion of a periodic function.** Given a periodic function $x : \mathbb{R} \to \mathbb{R}$, that is, there is a number p such that

  $$x(t+p) = x(t), \quad \forall t \in \mathbb{R},$$

  under some conditions on $x(t)$ it can be expressed as a (generally infinite) sum of sinus and cosinus:

  $$x(t) = \sum_{k=1}^{\infty} \big(a_k \cos(2\pi tk/p) + b_k \cos(2\pi tk/p)\big)$$

  The series on the right hand side is called the Fourier series expansion of f. This was an extraordinary result that have many applications in both theoretical and applied mathematics.

- **Fourier transform.** The Fourier transform of an integrable function $x : \mathbb{R} \to \mathbb{R}$, is the function

  $$\varphi : \mathbb{R} \to \mathbb{C}$$

  defined by

  $$\varphi(s) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi ts}\, dt,$$

  Under some conditions, the function $x(t)$ can be reconstructed from $\varphi(s)$ applying the inverse Fourier transform

  $$x(t) = \int_{-\infty}^{\infty} \varphi(s) e^{i2\pi ts}\, dt.$$

  In probability theory, there is the equivalent notion of the characteristic function of a random variable. Again, these results form a masterpiece in Mathematics, and many difficult problems couldn't be solved until that theory was on service.

- **Discrete Fourier Tranform.** The discrete Fourier transform (**DFT**) of a sequence of N real or complex numbers, $x_0, \ldots, x_{N-1}$, is the sequence of complex numbers $c_0, \ldots, c_{N-1}$ defined by

$$c_j = \frac{1}{N} \sum_{n=0}^{N-1} x_n e^{-2\pi ijn/N},\ j = 0, \ldots, N-1.$$

The original numbers $x_0, \ldots, x_{N-1}$ can be recovered from its DFT, through the so-called Inverse Discrete Furier Transform (**IDFT**)

$$x_j = \sum_{n=0}^{N-1} c_n e^{2\pi ijn/N},\ j = 0, \ldots, N-1.$$

So we have a bijection

$$x_0, \ldots, x_{N-1} \quad \longleftrightarrow \quad c_0, \ldots, c_{n-1}$$

This apparently innocent property has many applications, specially after the discovering in the 1960s of an efficient algorithm to compute the DFT for large sequences of numbers, called Fast Fourier Transform (**FFT**). The FFT is a topic by itself and we do not study it.

2 Continuous periodic functions

We will only consider discrete time series. However, it is convenient to comment the concept of periodic function in continuous time. Let x be a real function

$$\begin{aligned} x : \mathbb{R} &\longrightarrow \mathbb{R} \\ t &\longmapsto x(t) \end{aligned}$$

We say that $\{x(t),\, t \in \mathbb{R}\}$ is periodic with period p if

$$x(t+p) = x(t), \quad \forall t \in \mathbb{R},$$

and p is the smallest positive number that satisfies that condition. The addition of two functions with periods p_1 and p_2 such that p_1/p_2 is rational number is also a periodic function.

The prototype of a periodic fucntion is the sinus function: $x(t) = \sin\, t$ which is periodic with period 2π. More generally, for any $\mu > 0$, the function

$$x(t) = \sin(\mu t)$$

has period $p = 2\pi/\lambda$. To facilitate the interpretation, it is more convenient to write the function as

$$x(t) = \sin(2\pi\lambda t)$$

that has period $1/\mu$ and frequency λ. This λ is measured in cicles/unit of time, and it counts the number of times

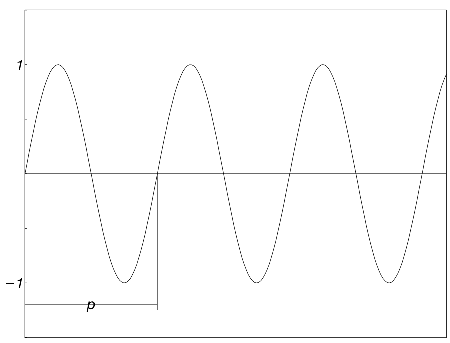


Figure 1. $x(t) = \sin(2\pi\lambda t)$

that all the structure of the function goes away for unit of time. If the time is measured in seconds, then the frequency is measured in cicles/second= 1 Hertz. Usually, Hertz is the unit used in Physics. That frequency λ (also called ***circular frequency***) should not be confused with the ***radial frequency***=$2\pi\lambda$ used in Mathematics, and that is measured in radians/unit of time.

Since $\sin(-x) = -\sin(x)$ we can also consider (non intuitive) negative frequencies.

Consider now $x(t) = b\sin(2\pi\lambda t)$. The period is again $p = 1/\lambda$. The number b is called the **amplitude**; the function varies between $-b$ and b. Finally, a **sinusoid** or sine wave is

$$x(t) = b\sin\big(2\pi(\lambda t + \varphi)\big),$$

where φ is called the **phase** and shifts left the function $bsin(2\pi\lambda t)$ the amount φ/λ. Note that

$$b\sin\big(2\pi(\lambda t + \varphi)\big) = a\cos(2\pi\lambda t) + c\sin(2\pi\lambda t).$$

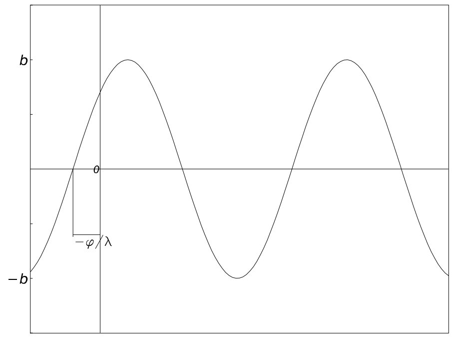
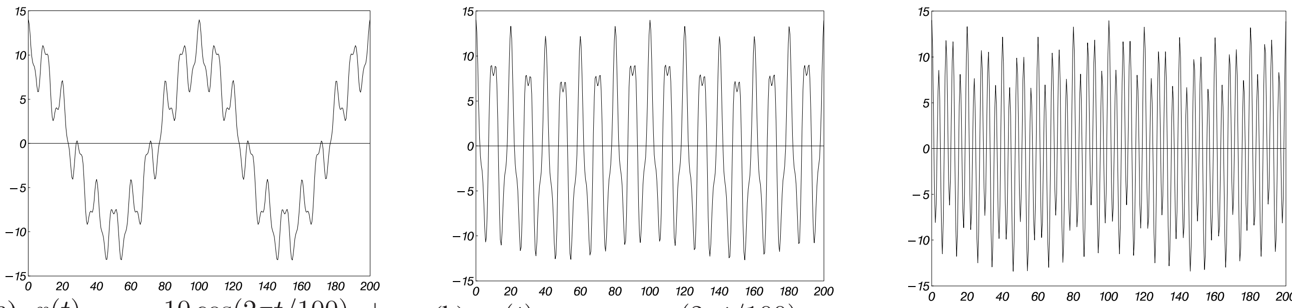


Figure 2. $b\ \sin\big(2\pi(\lambda t + \varphi)\big)$

Adding sinus and cosinus of different frequencies (satisfying the condition that the quocient of frequencies is a rational number) we get periodic functions with very diferent shapes (see the Figure 3).



(a) $x(t) = 10\cos(2\pi t/100) + 3\cos(2\pi t/10) + \cos(2\pi t/4)$

(b) $x(t) = \cos(2\pi t/100) + 10\cos(2\pi t/10) + 3\cos(2\pi t/4)$

(c) $x(t) = \cos(2\pi t/100) + 3\cos(2\pi t/10) + 10\cos(2\pi t/4)$

Figure 3. Periodic functions

The fundamental result of Fourier Analysis is that every periodic function of period p *enough regular* can be written in a sum (possibly, infinite) called *Fourier series* or *spectral representation*:

$$x(t) = \sum_{k=0}^{\infty}\big(a_k\cos(2\pi tk/p) + b_k\sin(2\pi kt/p)\big)\,,$$

and $\dfrac{k}{p}$ are called the ***Fourier frequencies***. The numbers a_k and b_k are given by

$$a_k = \frac{2}{p}\int_{-p/2}^{p/2} x(t)\cos(2\pi\lambda_k t)\,dt$$

and

$$b_k = \frac{2}{p}\int_{-p/2}^{p/2} x(t)\sin(2\pi\lambda_k t)\,dt$$

The frequency $1/p$ is called ***fundamental frequency*** , and the other frequencies $k/p,\ k \geq 2$ are called the ***harmonics***. This representation is the mathematical version of the decomposition of light in its monochromatic factors: each color corresponds to a frequency.

Using the complex exponential representation of the *sinus* and *cosinus*

$$\sin\theta = \frac{1}{2i}\big(e^{i\theta} - e^{-i\theta}\big)$$
$$\cos\theta = \frac{1}{2}\big(e^{i\theta} + e^{-i\theta}\big)$$

we deduce

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \exp\{i2\pi tk/p\}, \tag{1}$$

where

$$c_0 = a_0,$$
$$c_k = \frac{a_k - ib_k}{2}, \quad j \geq 1,$$
$$c_j = c_{-j}^* = \frac{a_j + ib_j}{2}, \quad j \leq -1.$$

where c^* denotes the complex conjugate of c.

Note that the condition $c_j = c_{-j}^*$ implies that the sum of the series is a real number.

Remark. From now on, $|c|$ will denote the absolute value or the module of c depending if it is a real or complex number.

2.1 Fourier analysis of a discrete periodic function

All previous analysis can be repeated for a function $x : \mathbb{Z} \to \mathbb{R}$. Assume that our function is the restriction to $\mathbb{Z}$ of a function on $\mathbb{R}$; this is a very standard setup: the phenomena happens in continuous time but we observe in discrete time; it is said that the continuous signal is **sampled** at discrete units of time Δt. So we have $x_1, x_2, \ldots,$ at times $\Delta t,\ 2\Delta t$. Then happens a very interesting fact that we study on an example:

Consider the functions

$$x(t) = \sin\Big(\frac{2\pi t}{3}\Big) \quad \text{and} \quad y(t) = \sin\Big(\frac{8\pi t}{3}\Big).$$

They coincide in the integer points $\forall n \in \mathbb{Z}$,

$$y(n) = \sin(2\pi n + 2\pi n/3) = \sin(2\pi n/3) = x(n).$$

The function $y(t)$, has frequency $\lambda = 8\pi/3 > \pi$, has oscillations between two consecutive integers n and $n+1$ that we cannot observe if we only have the values $y(n)$, for $n \in \mathbb{Z}$.

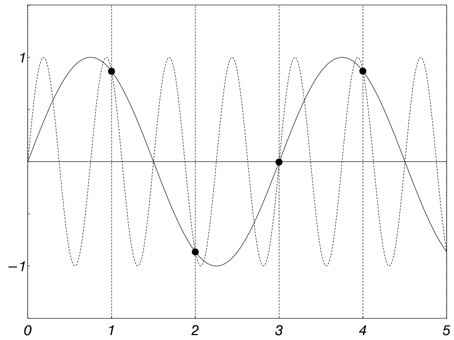


Figure 4. Aliasing phenomena

Return to the general case: based on a function observed at the integer numbers only the frequencies $\lambda \leq 0.5$; can be recorded; more specifically, each frequency $\lambda > 0.5$ has a frequency in $[0, 0.5]$ (called the **alias**) witch gives the same values of the function at the points $n \in \mathbb{Z}$. This phenomena is called *alialiasing*. By this reason, the spectral representation of a periodic discrete function of period p, $\{x(n), n \in \mathbb{Z}\}$ only includes frequencies $\lambda_k = \dfrac{k}{p} \in [0, 0.5]$.

3 Discrete Fourier Transform

Consider N numbers (real or complex), $x_0, \dots, x_{N-1}$, indexed by convenience by $0, \dots, N-1$. We can construct the periodic sequence (in principle, of period N)

$$\dots, x_0, \dots, x_{N-1},\ x_0, \dots, x_{N-1}, \dots \tag{2}$$

and apply a similar reasoning as in the continuous case. However, it is easier to proceed directly. The Discrete Fourier Transform, **DFT** henceforth, of $x_0, \dots, x_{N-1}$ is the sequence of complex numbers $c_0, \dots, c_{N-1}$ defined by

$$\boxed{c_j = \frac{1}{N}\sum_{n=0}^{N-1} x_n e^{-2\pi i j n/N},\ j = 0 \dots, N-1.} \tag{3}$$

We will prove later that the data $x_0, \dots, x_{N-1}$ can be recovered from its DFT $c_0, \dots, c_{N-1}$ applying the Inverse Discrete Fourier Transform (IDFT),

$$\boxed{x_j = \sum_{n=0}^{N-1} c_n e^{2\pi i j n/N},\ j = 0 \dots, N-1.} \tag{4}$$

At this point to gain a bit of intuition, it is convenient to practice. Remember the formula

$$e^{i\theta} = \cos\theta + i\sin\theta,\ \theta \in \mathbb{R}.$$

Compute the DFT of the following sequences (after the first one, the other are immediate):

(a) 1, 2, 3, 4.

(b) 1, 2, 1, 2.

(c) 1, 1, 1, 1.

(d) 1, 0, 0, 0.

In practice, the DFT is computed using the Fast Fourier Transform (FFT) that is an algorithm (indeed a family of algorithms). Here there is subtle distinction between DFT, that is a mathematical concept, and FFT that is a way to compute the DFT; anyway, it is important to know that the FFT gives the right value of a DFT. In **R** program there is the instruction `fft(x)` where `x` is a vector, however it does not divide by N, so in agreement with our definitions, we use the instruction

```
fft(x)/length(x)
```

The IDFT is obtained with `fft(x,inverse=T)`.

Remarks.

1. There is no total agreement about where to put the factor $1/N$, in (3) or in (4). There are still other possibilities, since the important fact is to put constants in (3) and (4) such that their product is equal to $1/N$. Some authors use $1/\sqrt{N}$ in each formula, that has some advantages. I have choose the factor that seems more suitable for time series analysis.

2. The sequence $c_j, j = 0, \dots, N-1$ can be extended to all $\mathbb{Z}$ with the same definition:

$$c_j = \frac{1}{N} \sum_{n=0}^{N-1} x_n e^{-2\pi i j n/N}, \ j \in \mathbb{Z}.$$

This sequence has period N. From that periodicity if follows that

$$c_{N-j} = c_{-j}.$$

3. A traditional notation in Fourier analysis is to write $W_N = e^{-2\pi i/N}$, which is a N–root of unity. Since the number N is fixed in all chapter, we suppress the subindex to short the notation. The following relations are trivial but very important (please, check!):

$$W^N = 1, \quad W^* = W^{-1}, \quad W^{j+k} = W^j\, W^k, \quad \text{and} \quad W^{N-j} = W^{-j}.$$

(Remember, W^* is the complex conjugate of W). With this notation,

$$c_j = \frac{1}{N} \sum_{n=0}^{N-1} x_n W^{nj}. \tag{5}$$

The periodicity of c_j is obvious from such expression.

4. From the previous points, it is deduced that an alternative expression for the IDFT is

$$x_j = \sum_{n=-[(N-1)/2]}^{[N/2]} c_n e^{2\pi ijn/N},$$

which appears in some books and looks more coherent with (1).

5. This property is very important:

$$\boxed{x_0, \dots x_{N-1} \in \mathbb{R} \quad \Longrightarrow \quad c_{N-j} = c_j{}^*}$$

This is easily deduced from the expression (5) and the properties of W.

3.1 Vectorial notations

We write the vectors in column. Let $\boldsymbol{x} = (x_0, \dots, x_{N-1})^t$ and $\boldsymbol{c} = (c_0, \dots, c_{N-1})^t$ its DFT, where $\boldsymbol{a}^t$ means the transposed of a vector or matrix $\boldsymbol{a}$. Set

$$\boldsymbol{F} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W & W^2 & \cdots & W^{N-1} \\ 1 & W^2 & W^4 & \cdots & W^{2(N-1)} \\ \vdots & & & & \vdots \\ 1 & W^{N-1} & W^{2(N-1)} & \dots & W^{(N-1)^2} \end{pmatrix}$$

which is a Vandermonde matrix, that has non zero determinant. The formula for the DFT (5) can be written as

$$\boldsymbol{c} = \frac{1}{N}\boldsymbol{F}\boldsymbol{x}. \tag{6}$$

Since det$\boldsymbol{F}$ is non zero, the application

$$\mathbb{C}^N \longrightarrow \mathbb{C}^N$$
$$\boldsymbol{x} \mapsto \boldsymbol{c} = \frac{1}{N}\boldsymbol{F}\boldsymbol{x}$$

is a bijection. Indeed, we see that DFT (and IDFT) is a linear map (as vector transforms).

Denote by $\boldsymbol{a}^*$ the conjugate transposed of the vector or matrix $\boldsymbol{a}$. Then

$$\boldsymbol{F}\boldsymbol{F}^* = N\boldsymbol{I}. \tag{7}$$

This is proved using the key property that if z is a N root of the unity different from 1, then

$$1 + z + z^2 + \cdots + z^{N-1} = \sum_{n=0}^{N-1} z^n = \frac{z^N - 1}{z - 1} = 0,$$

by the formula of the sum of a geometric sequence.

From (7) it follows

$$\boldsymbol{F}^{-1} = \frac{1}{N}\boldsymbol{F}^*,$$

so, inverting (6),

$$\boldsymbol{x} = N\boldsymbol{F}^{-1}\boldsymbol{c} = \boldsymbol{F}^{*}\boldsymbol{c},$$

which is the formula (4) for IDFT.

Write

$$\boldsymbol{e}_n = \big(1, W^{-n}, W^{-2n}, \dots, W^{-(N-1)n}\big)^t, \; n = 0, \dots, N-1,$$

and note that

$$(\boldsymbol{e}_0, \dots, \boldsymbol{e}_{N-1}) = \boldsymbol{F}^{*},$$

so the determinant of these vectors is non zero, and this implies that they form a basis of $\mathbb{C}^n$. Moreover, as before it is proved that

$$\boldsymbol{e}_n\, \boldsymbol{e}_m^{*} = N\, \delta_{nm}.$$

In the light of these results, the IDFT is nothing more (and nothing less!) that the expression of the vector $\boldsymbol{x}$ in the orthogonal basis $\boldsymbol{e_0, \dots, e_{N-1}}$.

Moreover,

$$\sum_{j=0}^{N-1} |c_j|^2 = \boldsymbol{c}\,\boldsymbol{c}^{*} = \boldsymbol{x}\boldsymbol{F}\boldsymbol{F}^{*}\boldsymbol{x}^{*} = \frac{1}{N}\sum_{j=0}^{N-1} |x_j|^2,$$

so we have the Parseval's Theorem for DFT,

$$\|\boldsymbol{c}\|^2 = \frac{1}{N}\,\|\boldsymbol{c}\|^2. \tag{8}$$

This formula have important applications.

3.2 The sinusoidal expression or the IDFT.

From now on, we will assume that $\boldsymbol{x_0, \dots, x_{N-1}}$ are real numbers. So $\boldsymbol{c_{N-j} = c_j{}^{*}}$.

The expression of the IDFT using complex exponential has the advantage of its analytical simplicity; however, to have an intuitive interpretation it is better to rewrite using sinusoids. It is needed to separate the case N even or odd.

Using the formula

$$e^{i\theta} = \cos\theta + i\sin\theta, \; \theta \in \mathbb{R}$$

and with the help of the properties of the trigonometric functions (in particular that $\cos 0 = 1$, $\sin 0 = 0$, $\cos(\pi j) = (-1)^j$, and $\sin(\pi j) = 0$), we get the following representations:

$$x_j = a_0 + \sum_{n=1}^{N/2-1} \big(a_n \cos(2\pi jn/N) + b_n \sin(2\pi jn/N)\big) + a_{N/2}(-1)^j, \quad \text{if } N \text{ is even,}$$

and

$$x_j = a_0 + \sum_{n=1}^{(N-1)/2} \big(a_n \cos(2\pi jn/N) + b_n \sin(2\pi jn/N)\big), \quad \text{if } N \text{ is odd,}$$

where,

$$a_0 = c_0,\ a_n = 2\Re(c_n),\ b_n = -2\Im(c_n), n = 1, \ldots, \frac{N}{2} - 1, \quad \text{and} \quad a_{N/2} = \Re(c_{N/2}), \quad \text{if } N \text{ is even,}$$

and

$$a_0 = c_0,\ a_n = 2\Re(c_n) \quad \text{and} \quad b_n = -2\Im(c_n), n = 1, \ldots, \frac{N-1}{2}, \quad \text{if } N \text{ is odd.}$$

Note that whether N is even or odd there are in total N parameters a_n and b_n in these expressions.

As a consequence, the IDFT gives the representation of the sequence of numbers $x_0, \ldots, x_{N-1}$ as a superposition of sinusoids with frequencies $0, 1/N, \ldots, [N/2]/N$, that are called the Fourier frequencies.

3.3 Computing the sinusoids

Of course, de easiest way to compute the coefficients a_n and b_n is using the DFT and the previous formulas. An alternative way is by linear regression. The unknown a_n and b_n enter linearly in the expression of x_j, so we have in hands a linear model. Since there are N numbers $(x_0, \ldots, x_{N-1})$ and we want to compute N numbers (the a_n and b_n) the fit is perfect and there is zero degrees of freedom. The independent variables are $\cos(2\pi jn/N)$ and $\sin(2\pi jn/N)$.

3.4 The periodogram

Write

$$I_j = \frac{N}{2}|c_j|^2,\ j = 0, \ldots, [N/2].$$

The plot of the points $(j/N, I_j)$ for j as above, is called the **periodogram**. Note that $j/N \in [0, 0.5]$, in agreement with the aliasing phenomena. Normally there are strong variations between de different frequencies, and then the periodogram is plotted in a vertical logarithmic scale.

In Table 1 from Newton [5] there are some interesting comments about the appearance of a periodogram

Table 1. Interpreting the periodogram

Appearance of the data	Nature of periodogram
Smooth	Excess of low frequency; that is, amplitudes of sinusoids of low frequency (long period) are large relative to other frequencies
Wiggly	Excess of high frequency
No patern	No frequencies dominate
Basically sinusoidal of period p	A peak at frequency $1/p$
Periodic of period p but not sinusoidal	A peak at fundamental frequency $1/p$ and peaks at some multiples of $1/p$ (harmonics)

3.5 Periodic sequences

Assume that $x_0, \ldots, x_{N-1}$ is periodic of period ℓ, and that N is multiple of ℓ. Then it is proved that $c_j = 0$ except for $j = \ell, 2\ell, \ldots$. Interpret this property in terms of the decomposition in a sum of sinusoids.

3.6 DFT and the decomposition of the variance

Write

$$\overline{x} = \frac{1}{N} \sum_{n=0}^{N-1} x_n$$

the mean of the numbers $x_0, \ldots, x_{N-1}$, and s^2 its variance:

$$s^2 = \frac{1}{N} \sum_{n=0}^{N-1} \left(x_n - \overline{x}\right)^2 = \frac{1}{N} \sum_{n=0}^{N-1} x_n^2 - (\overline{x})^2.$$

First, note that

$$\boxed{c_0 = \overline{x}.}$$

Second, by Parseval's Theorem (8),

$$s^2 = \sum_{n=1}^{N-1} |c_n|^2.$$

Since $c_{N-j} = c_j^*$, the terms in the sum appear repited, so

$$s^2 = \begin{cases} 2\sum_{n=1}^{[N/2]} |c_n|^2, & \text{if } N \text{ is odd,} \\ \\ 2\sum_{k=1}^{N/2-1} |c_n|^2 + |c_{(N/2)}|^2, & \text{if } N \text{ is even.} \end{cases}$$

The following table is a type of decomposition of the variance when N is even (Cowpertwait and Metcalfe [3, page 174]):

Harmonic	Period	Frequency	Contribution to variance
1	N	$1/N$	$2\|c_1\|^2$
2	$N/2$	$2/N$	$2\|c_2\|^2$
$\vdots$			
$N/2-1$	$N/(N/2-1)$	$(N/2-1)/N$	$2\|c_{(N/2-1)}\|^2$
$N/2$	2	$1/2$	$\|c_{N/2}\|^2$

Exercices

1. Compute (with **R**) the DFT of the sequences

(a) 1, 2, 3, 4.

(b) 1, 2, 1, 2.

(c) 1, 1, 1, 1.

2. Plot the periodograms of the previous sequences.

3. Using linear regression, compute the expression with sinusoids of the sequence 1, 2, 3, 4. First, write the linear model as

$$\begin{aligned} x_0 &= a_0 + a_1 \cos(2\pi 1 \cdot 0/4) + b_1 \sin(2\pi 1 \cdot 0/4) + a_2 \cos(2\pi 2 \cdot 0/4) \\ x_1 &= \cdots \\ &\vdots \end{aligned}$$

The independent variables are $\cos(2\pi jn/4)$ and $\sin(2\pi jn/4)$, so you need to prepare a matrix with that variables, called, for example, `A`, and a vector with the values $(x_0, \ldots, x_3)$, called, say, `x`. The instruction to fit a linear model (with intercept) is

```
lm(x~A)
```

Check the relations given in the text between c_j and a_j and b_j.

4. Using the numbers a_0, a_1, b_1, a_2 of the previous exercise, consider the functions

$$\begin{aligned} a_0(t) &= a_0, \\ c_1(t) &= a_1 \cos(2\pi t/4) \\ s_1(t) &= a_1 \sin(2\pi t/4) \\ c_2(t) &= a_2 \cos(2\pi t/4) \end{aligned}$$

In the same graphic, plot the points $(0,1), (1,2), (2,3), (3,4)$ and the functions (in continuous time) $a_0(t),\ c_1(t), \ldots$. Convince yourself that

$$x_j = a_0(j) + c_1(j) + s_1(j) + c_2(j),\ j = 0, \ldots, 3.$$

Repite the exercise with the sequences (b) and (c).

5. Give a function that in the integer values coincides with the function

$$x(t) = 2\cos(2\pi t/100) + 6\sin(2\pi t/20).$$

Plot both functions in the same graphic.

6. Plot the periodogram of the sequences corresponding to Figure 3:

- $x(j) = 10\cos(2\pi j/100) + 3\cos(2\pi j/10) + \cos(2\pi j/4),\ j = 0, \ldots, 199.$
- $x(j) = \cos(2\pi j/100) + 10\cos(2\pi j/10) + 3\cos(2\pi j/4),\ j = 0, \ldots, 199.$
- $x(j) = \cos(2\pi j/100) + 3\cos(2\pi j/10) + 10\cos(2\pi j/4),\ j = 0, \ldots, 199.$

Note that you can use the formulas given in this notes, or compute the FFT. Contrast the visual information with the comments on Table 1.

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