Stochastic models in population dynamics

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July 24, 2009

We consider a Markov chain, $(X_n, n \geq 0)$, in which there is a set T of transient states from which the process is certain to be absorbed into the remaining states. Let's suppose the chain have states $0, 1, 2, ..., s$ with transition matrix

$$P = \begin{bmatrix} 1 & 0' \\ p_0 & Q \end{bmatrix}$$

$p_0 \neq 0$, Q is $s \times s$ irreducible, aperiodic substochastic matrix and $p_0, 0$ are $s \times 1$ vectors.

Definition Let $[q_0(n), q'(n)]$ denote the probability distribution of X_n over all states (s+1) states at time n and denote by $d(n)$ the conditional distribution

$$d(n) = \frac{q(n)}{1 - q_0(n)}.$$

Equivalently, we can say $d_k(n) = P_{d(0)}(X_n = k | X_n \neq 0)$
If $d(n+1) = d(n) = d$ then, we call d a quasi stationary distribution. $P_d(X_n = k | X_n \neq 0) = d_k$.

Proposition d is a quasi-stationary distribution if and only if d is the left eigenvector of Q with non-negative components.

Proposition Conditional distribution of discrete time finite Markov chain, starting from an arbitrary initial distribution π, converges to the quasi-stationary distribution.

$\triangleleft$ The conditional probability that it is in state j at time n is

$$P_\pi(X_n = j | X_n \neq 0) = \frac{\sum_{i \in T} \pi_i p_{ij}^{(n)}}{\sum_{i \in T} \pi_i (1 - p_{i0}^{(n)})} = \frac{\pi' Q^n f_j}{\pi' Q^n e}$$

where f_j is $s \times 1$ vector with unity in the j-th row and zeros elsewhere.
Since Q is irreducible and aperiodic, by Perron Frobenius theorem $Q^n = \rho_1^n \omega v' + o(n^k |\rho_2|^n)$

$$B = \frac{\sum_{i \in T} \pi_i p_{ij}^{(n)}}{\sum_{i \in T} \pi_i (1 - p_{i0}^{(n)})} = v_j + o(n^k (\frac{|\rho_2|}{\rho_1})^n).$$

$B \longrightarrow v_j$ as $n \longrightarrow \infty$. $\triangleright$

Let's consider a birth and death chain in $[0,..,k]^2$ with the following transition:
1) $(i,j) \to (i,j)$ w.p. r_{ij},
$(i,j) \to (i+1,j)$ w.p. $p_{ij}^{(1)}$, $(i,j) \to (i,j+1)$ w.p. $p_{ij}^{(2)}$
$(i,j) \to (i-1,j)$ w.p. $q_{ij}^{(1)}$, $(i,j) \to (i-1,j)$ w.p. $q_{ij}^{(2)}$.
2) (0,0) is only absorbing state.
3) When either coordinate hits zero there is no birth and death probability for that component and when either coordinate reaches state "k" then there is no birth probability.

When we order our states appropriately as (0,0), (0,i), (i,0), (i,j) $\forall 1 \leq i, j \leq k$, we will have the following stochastic matrix:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & A_1 & 0 & 0 \\ * & 0 & A_2 & 0 \\ 0 & B & C & A_3 \end{bmatrix}$$

We want to find $\lim_{n\to\infty} P_{(a,b)}((X_n, Y_n) = (i,j) | (X_n, Y_n) \neq 0)$

Let's denote $\lambda 1, \lambda 2, \lambda 3$ as maximal eigenvalue of A_1, A_2, A_3 and S_1, S_2, S_3 as corresponding states for matrixes.
$\lambda = \max\{\lambda 1, \lambda 2, \lambda 3\}, v_i, \omega_i$ are the positive left and right eigenvectors of A_i.
1) Let's suppose $(a,b),(i,j) \in S_3$ and if the $\lambda 3 \neq \lambda$ then

$$\lim_{n\to\infty} P_{(a,b)}((X_n, Y_n) = (i,j) | (X_n, Y_n) \neq 0) = 0.$$

2) if $(a,b),(i,j) \in S_3, \lambda 3 = \lambda > \lambda 1, \lambda 2$ then

$$\lim_{n\to\infty} P_{(a,b)}((X_n, Y_n) = (i,j) | (X_n, Y_n) \neq 0) = \frac{v_3'(ij)}{1+v_3' B(\lambda 3 I - A_1)^{-1} e + v_3' C(\lambda 3 I - A_2)^{-1} e}$$

3) if $(a,b) \in S_3, (i,j) \in S_2, \lambda 3 = \lambda > \lambda 1, \lambda 2$ then

$$\lim_{n\to\infty} P_{(a,b)}((X_n, Y_n) = (i,j) | (X_n, Y_n) \neq 0) = \frac{v_3' C(\lambda 3 I - A_2)^{-1} f_{ij}}{1+v_3'(\lambda 3 I - A_1)^{-1} e + v_3' C(\lambda 3 I - A_2)^{-1} e}$$

4) If $(a, b) \in S_3, (i, j) \in S_2, \lambda 2 = \lambda > \lambda 1, \lambda 3$ then

$$\lim_{n\to\infty} P_{(a,b)}((X_n, Y_n) = (i, j)|(X_n, Y_n) \neq 0) = v_2(ij)$$

5) If $(a, b) \in S_3, (i, j) \in S_2, \lambda = \lambda 1 = \lambda 2 > \lambda 3$ then

$$\lim_{n\to\infty} P_{(a,b)}((X_n, Y_n) = (i, j)|(X_n, Y_n) \neq 0) = \frac{f'(ab)(\lambda I - A_3)^{-1} C\omega_2}{f'(ab)(\lambda I - A_3)^{-1} C\omega_2 + f'(ab)(\lambda I - A_3)^{-1} B\omega_1}$$

6) When $\lambda 1 = \lambda 2 = \lambda 3$,

$$\lim_{n\to\infty} P_{(a,b)}((X_n, Y_n) = (i, j)|(X_n, Y_n) \neq 0) = \begin{cases} pv_1(ij) \ , \ (ij) \in S_1 \\ (1-p)v_2(ij) \ , \ (ij) \in S_2 \end{cases} \text{ where } p = \frac{v_3' B\omega_1}{v_3'(C\omega_2 + B\omega_1)}$$

Neutral birth and death model

We consider state space as $(ij) = \{0 \leq i + j \leq N\}$ with following transition matrix: $p^1_{ij} = \frac{i}{i+j}\lambda(i+j), p^2_{ij} = \frac{j}{i+j}\lambda(i+j)$, $q^1_{ij} = \frac{i}{i+j}\mu(i+j), q^2_{ij} = \frac{j}{i+j}\mu(i+j)$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & A & 0 & 0 \\ * & 0 & A & 0 \\ 0 & B & C & A_1 \end{bmatrix}$$

We can prove that P has the right eigenvectors of the form $\{v_{i+j}\}, \{iv_{i+j}\}, \{jv_{i+j}\}, \ldots$

Proposition If

$$\begin{cases} iP_d(i+1,j) + jP_d(i,j+1) = (i+j+d)P_d(i,j) \\ iP_d(i-1,j) + jP_d(i,j-1) = (i+j-d)P_d(i,j) \end{cases}$$

$\deg(P_d) = d$ then P has the eigenvectors of the form $P_d(i,j)v_{i+j}$.
$\triangleleft$ $A_1 : \mathbb{R}^d \to \mathbb{R}^d$ where $d = N(N+1)/2$,
$u \in \mathbb{R}^d, u = (u_{ij}, (ij) \in S)$, for $(i,j) \neq 0$
$(Pu)_{ij} =$
$\frac{i}{i+j}\lambda_{i+j}u_{i+1,j} + \frac{j}{i+j}\lambda_{i+j}u_{i,j+1} + \frac{i}{i+j}\mu_{i+j}u_{i-1,j} + \frac{j}{i+j}\mu_{i+j}u_{i,j-1}$,
$\triangleright$
Therefore we can find more forms of eigenvectors by finding polynomials which satisfy above proposition.
For example: $P_2 = ij, P_3 = ij(i-j)$ (symmetric),
$P_4 = i^3j + ij^3 - 3i^2j^2 + ij$

Particle method:
1) At the same time we run n particles, when it becomes extinct we randomly replace it with another existing particle.

2) At the same time we run n particles and each time every particle is chosen randomly from the previous step extant particles.

The semi-infinite simple random walk with absorbtion The substochastic matrix Q which describes the simple random walk on the non-absorbing states $T = \{1, 2, ...\}$ is irreducible and periodic with period 2.

$$Q = \begin{bmatrix} 0 & b & 0 & 0 & \\ a & 0 & b & 0 & \ddots \\ 0 & a & 0 & b & \\ & \ddots & & \ddots & \end{bmatrix}$$

$$v_j = v_1 j \left(\sqrt{\frac{b}{a}}\right)^{j-1}, j = 1, 2... \text{ and } \omega_j = \omega_1 j \left(\sqrt{\frac{a}{b}}\right)^{j-1}, j = 1, 2...$$

$$\lim_{n \to \infty} \frac{\sum_{m=1}^{n} v_{ij}^{(m)}}{n} = c_j j \left(\sqrt{\frac{b}{a}}\right)^{j-1} = c_j v_j$$

$$c_j = \begin{cases} 0 \text{ ,if } a = b = 1/2 \\ \frac{1}{2}[\frac{1-4ab}{a}] \text{ ,if } a > b, j \text{ odd} \\ \frac{1}{2}[\frac{1-4ab}{a}\frac{1}{2\sqrt{ab}}] \text{ , if } a > b, j \text{ even} \end{cases}$$

The result of simulation pi2 - numerical value, pi1 - analytical solution.

$$\text{pi2} = 0.0134$$
$$\text{pi1} = 0.0141707$$
$$\text{err} = 0.0007707$$

Bonus When we to allow to pause in any state, we will obtain more general matrix of the following form: (the matrix consists of non-absorbing states $T = \{1, 2, ...\}$, excluding only absorbing state $\{0\}$)

$$Q1 = \begin{bmatrix} c & b & 0 & 0 & \\ a & c & b & 0 & \ddots \\ 0 & a & c & b & \\ & \ddots & & \ddots & \end{bmatrix}$$

$$P_i(X_k = j | X_k \neq 0) \to v_j$$

where $\sum_{j=1}^{\infty} v_j = 1$, if $a > b$.

Thank you very much for attention.