Chemotaxis problem

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1 Introduction

The term chemotaxis indicates a biological phenomenon describing the change of motion when a population formed of individuals (such as amoebae, bacteria, endothelial cells...), reacts in response (taxis) to an external chemical stimulus spread in the environment where they reside. As a consequence, the population changes its movement toward (positive chemotaxis) higher concentration of the chemical substance. A possible fascinating issue of a positive chemotactical movement is the aggregation of the organisms involved to form a more complex organism or body. The basic feature of such phenomena is the presence of concentration effects possibly leading to non uniform pattern formation. The basic unknowns in PDE models for chemotaxis models are the density of individuals of the population and the concentration of the chemoattractant. The first and the most celebrated model of these phenomena is the Patlak-Keller-Segel model. In this case the basic assumption is that dynamic of individuals is described by a parabolic equation coupled with an additional equation for the chemoattractant, chosen to be elliptic or parabolic, depending on the different regimes to be described. A large amount of articles and studies analyzed the Patlak-Keller-Segel system, see for example [1], [2], [3]. Existence of stationary solutions for the parabolic Keller-Segel model has been studied for both the simplest model and the more general one in [4] and [5]. It's also well known that in two and three dimensions the solutions can blow-up in finite time, see for example [6]. However in one space dimension the global existence of solutions for general initial data has been shown by Osaki and Yagi in [7]. In this work we are considering a particular case of the following problem:

$$\begin{cases} \partial_t \rho - \mu \partial_{xx} \rho + \partial_x(\rho \chi(c) \partial_x c) - \gamma \rho = 0 \\ \epsilon \partial_t c - \nu \partial_{xx} c + \beta c - \alpha \rho = 0 \end{cases} \tag{1}$$

with $\rho(x,t) \in \Re^+$, $c(x,t) \in \Re^+$, for $t \geq 0$, $\Omega = (a,b) \subset \Re$

together with the suitable hypotheses on the parameters:

$\alpha, \beta, \gamma, \epsilon, \mu, \nu \geq 0$

and with the following boundary conditions:

$\frac{\partial \rho}{\partial n}(a) = \frac{\partial c}{\partial n}(b) = 0$

$\frac{\partial \rho_0}{\partial n}(a) = \frac{\partial c_0}{\partial n}(b) = 0$

$\rho(0, x) = \rho_0(x) \geq 0$, and $c(0, x) = c_0(x) \geq 0$.

The terms in the first equation of (1) include the diffusion of bacteria, chemotactic drift and division of bacteria. Instead the second equation expresses diffusion, production and died of attractant.
Naturally according to the choice of the parameters it's possible to have different study case with different results.
In this paper we want to focus our attention in the case in which $\gamma = 0$, since we are considering a specific time of observation and the function $\chi(c) = \chi$ constant. It's also well know that if we consider $\epsilon = 0$ we obtain an elliptic equation for c.
The aim of this work is to perform an analytical and numerical study of the system (1) with the particular assumption on the parameters.

2 Weak formulation

We consider smooth solutions, regular enough and we perform the following analysis

$$\begin{cases} \langle \partial_t \rho, \varphi \rangle - \mu \langle \partial_{xx} \rho, \varphi \rangle + \chi \langle \partial_x (\rho \partial_x c), \varphi \rangle = 0 \\ \epsilon \langle \partial_t c, \psi \rangle - \nu \langle \partial_{xx} c, \psi \rangle + \beta \langle c, \psi \rangle - \alpha \langle \rho, \psi \rangle = 0 \end{cases} \tag{2}$$

where $\langle , \rangle$ denotes the usual scalar product in L^2, namely $\langle , \rangle = \int_a^b uv dx$.

Integrating by parts we obtain:

$$\begin{cases} \int_a^b \partial_t \rho \varphi dx - \mu \varphi \partial_x \rho |_a^b + \mu \int_a^b \partial_x \rho \partial_x \varphi dx \\ + \chi \varphi \rho \partial_x c |_a^b - \chi \int_a^b \rho \partial_x c \partial_x \varphi dx = 0 \\ \epsilon \int_a^b \partial_t c \psi dx - \nu \psi \partial_x c |_a^b + \nu \int_a^b \partial_x c \partial \psi + \beta \int_a^b c \psi dx - \alpha \int_a^b \rho \psi dx = 0 \end{cases} \tag{3}$$

And finally using boundary conditions:

$$\begin{cases} \int_a^b \partial_t \rho \varphi dx + \mu \int_a^b \partial_x \rho \partial_x \varphi dx = \chi \int_\Omega \rho \partial_x c \partial_x \varphi dx \\ \epsilon \int_a^b \partial_t c \psi dx + \nu \int_a^b \partial_x c \partial_x \psi dx = \alpha \int_a^b \rho \psi dx - \beta \int_a^b c \psi dx \end{cases} \tag{4}$$

We can rewrite this expressions in the convenient way:

$$\begin{cases} \langle \partial_t \rho, \varphi \rangle + \mu \langle \partial_x \rho, \partial_x \varphi \rangle - \chi \langle \rho \partial_x c, \partial_x \varphi \rangle = 0 \\ \epsilon \langle \partial_t c, \psi \rangle + \nu \langle \partial_x c, \partial_\psi \rangle + \beta \langle c, \psi \rangle - \alpha \langle \rho, \psi \rangle = 0 \end{cases} \tag{5}$$

Therefore we consider the functional space H^1 and we seek $\rho, c \in H^1$ such that for all $\varphi, \psi \in H^1$.

3 Finite element method

Let $\{V_h\}_{h \geq 0}$ be a family of approximating subspace of H^1, consisting of piecewise polynomials, and let $\{T_h\}_{h \geq 0}$ be a partition of (a, b) made of intervals $(x_i, x_{i+1}), i = 1, ..., N$ with $h = max_i(x_{i+1} - x_i)$.

We assume T_h regular and define $V_h = \{\varphi, \psi \in H^1; \varphi|_T, \psi|_T \in P(T), T \in T_h\}$ with T a generic interval (x_i, x_{i+1}) and P the family of polynomial of degree k.
The finite element solution of (5) consist in seeking a solution $\rho_h, c_h \in V_h$ of the following problem:

$$\begin{cases} \langle \partial_t \rho_h, \varphi \rangle + \mu \langle \partial_x \rho_h, \partial_x \varphi \rangle - \langle \rho_h \chi \partial_x c_h, \partial_x \varphi \rangle = 0 \\ \epsilon \langle \partial_t c_h, \psi \rangle + \nu \langle \partial_x c_h, \partial_x \psi \rangle + \beta \langle c_h, \psi \rangle - \alpha \langle \rho_h, \psi \rangle = 0 \end{cases} \tag{6}$$

for all $\varphi, \psi \in V_h$.

4 Stability estimate

4.1 L^2-energy estimate

Following [8] we consider $\varphi = \rho_h \in V_h$ in the first equation of (6)

$$\langle \partial_t \rho_h, \rho_h \rangle + \mu \langle \partial_x \rho_h, \partial_x \rho_h \rangle - \chi \langle \rho_h \partial_x c_h, \partial_x \rho_h \rangle = 0 \tag{7}$$

and so we obtain:

$$\frac{1}{2} \frac{d}{dt} \|\rho_h\|^2_{L^2} + \mu \|\partial_x \rho_h\|_{L^2} - \chi \langle \rho_h \partial_x c_h, \partial_x \rho_h \rangle = 0 \tag{8}$$

Then considering $\psi = c_h$ in the second equation of (6) we have:

$$\epsilon \langle \partial_t c_h, c_h \rangle + \nu \langle \partial_x c_h, \partial_x c_h \rangle + \beta \langle c_h, c_h \rangle - \alpha \langle \rho_h, c_h \rangle = 0 \tag{9}$$

and

$$\frac{\epsilon}{2} \frac{d}{dt} \|c_h\|^2_{L^2} + \nu \|\partial_x c_h\|^2_{L^2} + \beta \|c_h\|^2_{L^2} - \alpha \langle \rho_h, c_h \rangle = 0 \tag{10}$$

We sum (8) and (10) to obtain

$$\frac{1}{2}\frac{d}{dt}(\|\rho_h\|_{L^2}^2 + \epsilon\|c_h\|_{L^2}^2) + \mu\|\partial_x\rho_h\|_{L^2}^2 + \nu\|\partial_x c_h\partial\|_{L^2}^2 + \beta\|c_h\|_{L^2}^2$$

$$= \chi\langle\rho_h\partial_x c_h, \partial_x\rho_h\rangle + \alpha\langle\rho_h, c_h\rangle \tag{11}$$

Using Cauchy-Schwartz and Young inequalities we have that

$$\alpha\langle\rho_h, c_h\rangle \le \alpha\|\rho_h\|_{L^2}^2\|c_h\|_{L^2}^2 \le \frac{\alpha}{2\delta_1}\|\rho_h\|_{L^2}^2 + \frac{\alpha\delta_1}{2}\|c_h\|_{L^2}^2$$

for $\delta_1 << 1$ and for the non linear term

$$\chi\langle\rho_h\partial_x c_h, \partial_x\rho_h\rangle \le \chi\|\rho_h\partial_x c_h\|_{L^2}^2\|\partial_x\rho_h\|_{L^2}^2 \le \frac{\chi}{2\delta_2}\|\rho_h\partial_x c_h\|_{L^2}^2 + \frac{\chi\delta_2}{2}\|\partial_x\rho_h\|_{L^2}^2$$

for $\delta_2 << 1$. Then we can rewrite the equation (11) as

$$\frac{1}{2}\frac{d}{dt}(\|\rho_h\|_{L^2}^2 + \epsilon\|c_h\|_{L^2}^2) + \left(\mu - \frac{\chi\delta_2}{2}\right)\|\partial_x\rho_h\|_{L^2}^2 + \nu\|\partial_x c_h\|_{L^2}^2 + \left(\beta - \frac{\alpha\delta_1}{2}\right)\|c_h\|_{L^2}^2$$

$$\le \frac{\chi}{2\delta_2}\|\rho_h\partial_x c_h\|_{L^2}^2 + \frac{\alpha}{2\delta_1}\|\rho_h\|_{L^2}^2$$

Since we can consider

$$\frac{\chi}{2\delta_2}\|\rho_h\partial_x c_h\|_{L^2}^2 \le \frac{\chi}{2\delta_2}\|\rho_h\|_{L^\infty}^2\|\partial_x c_h\|_{L^2}^2 \tag{12}$$

we obtain

$$\frac{1}{2}\frac{d}{dt}(\|\rho_h\|_{L^2}^2 + \epsilon\|c_h\|_{L^2}^2) + \left(\mu - \frac{\chi\delta_2}{2}\right)\|\partial_x\rho_h\|_{L^2}^2 + \left(\nu - \frac{\chi}{2\delta_2}\|\rho_h\|_{L^\infty}^2\right)\|\partial_x c_h\|_{L^2}^2$$

$$+ \left(\beta - \frac{\alpha\delta_1}{2}\right)\|c_h\|_{L^2}^2 \le \frac{\alpha}{2\delta_1}\|\rho_h\|_{L^2}^2 \tag{13}$$

it holds if

$$\|\rho_h\|_{L^\infty}^2 \le \frac{\nu 2\delta_2}{\chi} \tag{14}$$

To be sure that the solution doesn't blow up we consider it only for short time and initial data very small, so we assume that
$|\rho(0)|_{L^\infty} \le \epsilon$
with ϵ small enough and we define the set
$I = \{t : |\rho|_{L^\infty}(t) < \frac{2\nu\delta_2}{\chi}\}$.
For all $t \in I$ we have that (13) holds and this allows us to write

$$\frac{1}{2}\frac{d}{dt}(\|\rho_h\|_{L^2}^2 + \epsilon\|c_h\|_{L^2}^2) \le \frac{\alpha}{2\delta_1}(\|\rho_h\|_{L^2}^2 + \epsilon\|c_h\|_{L^2}^2) \tag{15}$$

Solving this inequality we have

$$\|\rho_h\|_{L^2}^2 + \epsilon\|c_h\|_{L^2}^2 \leq (\|\rho(0)\|_{L^2}^2 + \epsilon\|c(0)\|_{L^2}^2)exp^{\frac{\alpha t}{2\delta_1}} \tag{16}$$

Considering that

$$\|\rho_h\|_{L^\infty}^2 \leq \frac{1}{h}\|\rho_h\|_{L^2}^2 \leq \frac{1}{h}\|\rho_h\|_{L^2}^2 + \epsilon\|c_h\|_{L^2}^2 \leq \frac{1}{h}(\|\rho(0)\|_{L^2}^2 + \epsilon\|c(0)\|_{L^2}^2)exp^{\frac{\alpha t}{2\delta_1}} \tag{17}$$

using (14) we obtain the following time estimation for which the solution remain bounded

$$t < \ln\left(\frac{2\nu\delta_2}{\chi}h\frac{1}{\|\rho(0)\|_{L^2}^2 + \epsilon\|c(0)\|_{L^2}^2}\right)\frac{2\delta_1}{\alpha} \tag{18}$$

4.2 Error estimates and convergence results

Following [9] we consider the equations for the errors between the analytical and the numerical solution:

$$E_\rho = \rho - \rho_h \tag{19}$$

$$E_c = c - c_h \tag{20}$$

that is for all $\varphi, \psi \in V_h \subset H^1$

$$\begin{cases} \langle \partial_t E_\rho, \varphi\rangle + \mu\langle \partial_x E_\rho, \partial_x\varphi\rangle - \chi\langle \rho\partial_x c - \rho_h\partial_x c_h, \partial_x\varphi\rangle = 0, \\ \epsilon\langle \partial_t E_c, \psi\rangle + \nu\langle \partial_x E_c, \partial_x\psi\rangle + \beta\langle E_c, \psi\rangle - \alpha\langle E_\rho, \psi\rangle = 0 \end{cases} \tag{21}$$

In the following, we introduce the standard arguments to get error estimates for finite element methods, based on the idea of splitting the errors into two parts the consistency errors η_ρ, η_c and the stability errors θ_ρ, θ_c, respectively for the density and the concentration, namely:

$$E_\rho = \rho - \rho_h = (\rho - \Pi_h\rho) + (\Pi_h\rho - \rho_h) = \eta_\rho + \theta_\rho \tag{22}$$

$$E_c = c - c_h = (c - \Pi_h c) + (\Pi_h c - c_h) = \eta_c + \theta_c \tag{23}$$

where Π_h is the Elliptic Projection operator such that

$$\langle \partial_x \Pi_h v, \partial_x\xi\rangle = \langle \partial_x v, \partial_x\xi\rangle \tag{24}$$

for all $v \in H^1, \xi \in V_h$.
This operator allows us to write that for ρ and c smooth enough we have that η_ρ and η_c satisfy the approximation properties:

$$\|\eta_\rho\| + h^{\frac{1}{2}}\|\partial_x\eta_\rho\| \leq C^{st}h^r\|\rho\| \tag{25}$$

$$\|\partial_t\eta_\rho\| \leq C^{st}h^r\|\rho\| \tag{26}$$

for $\rho \in H^r$.
Naturally the same inequalities are valid for η_c only replacing ρ with c. What we want to do is to find an estimation for the stability error.

4.3 L^2 error estimates for the error

We start considering the weak formulation satisfied by the stability error θ_c, that is for all $\psi \in V_h$:

$$\epsilon\langle\partial_t\theta_c,\psi\rangle+\nu\langle\partial_x\theta_c,\partial_x\psi\rangle+\beta\langle\theta_c,\psi\rangle = -\epsilon\langle\partial_t\eta_c,\psi\rangle-\beta\langle\eta_c,\psi\rangle+\alpha\langle E_\rho,\psi\rangle \quad (27)$$

We take $\psi = \theta_c \in V_h$ in (27) to get:

$$\frac{1}{2}\frac{d}{dt}\|\theta_c\|^2+\nu\|\partial_x\theta_c\|^2+\beta\|\theta_c\|^2 = -\epsilon\langle\partial_t\eta_c,\theta_c\rangle-\beta\langle\eta_c,\theta_c\rangle+\alpha\langle E_\rho,\theta_c\rangle$$

$$\leq \epsilon\|\partial_t\eta_c\|\|\theta_c\|+\beta\|\eta_c\|\|\theta_c\|+\alpha\|\eta_\rho+\theta_\rho\|\|\theta_c\| \leq (\epsilon\|\partial_t\eta_c\|+\beta\|\eta_c\|+\alpha\|\eta_\rho+\theta_\rho\|)\|\theta_c\|$$

Let t^* such that $\|\theta_c(t^*)\| = max_{t\geq 0}\|\theta_c(t)\|$, we integrate in $[0,t^*]$ to get:

$$\frac{\epsilon}{2}\|\theta_c(t^*)\|+\nu\int_0^{t^*}\|\partial\theta_c(t)\|^2dt+\beta\int_0^{t^*}\|\theta_c(t)\|^2dt$$

$$\leq \frac{\epsilon}{2}\|\theta_c(0)\|^2+\int_0^{t^*}(\epsilon\|\partial_t\eta_c(t)\|+\beta\|\eta_c(t)\|+\alpha\|\eta_\rho(t)+\theta_\rho(t)\|)\|\theta_c(t)\|dt$$

$$\leq \left[\frac{\epsilon}{2}\|\theta_c(0)\|+\int_0^{t^*}(\epsilon\|\partial_t\eta_c(t)\|+\beta\|\eta_c(t)\|+\alpha\|\eta_\rho(t)+\theta_\rho(t)\|)dt\right]\|\theta_c(t^*)\|$$

In a first attempt we can deduce that:

$$\|\theta_c(t^*)\| \leq \|\theta_c(0)\|+C_1\int_0^{t^*}(\|\partial_t\eta_c(t)\|+\|\eta_c(t)\|+\|\eta_\rho(t)+\theta_\rho(t)\|)dt$$

with $C_1 = 2C_1(1,\frac{\beta}{\epsilon},\frac{\alpha}{\epsilon}) \geq 0$.
Finally, using (25) and (26) we can conclude that

$$\|\theta_c(t)\| \leq \|\theta_c(0)+C_1C^{st}h^r\int_0^{t^*}(\|\partial_t c\|+\|c\|)dt+C_1\int_0^{t^*}\|\eta_\rho(t)+\theta_\rho(t)\|dt$$

$$\leq Ch^r+C_1\int_0^t\|(\rho-\rho_h)(s)\|ds \quad (28)$$

where $C = C(t, c_0, c_h(0), C_1, C^{st}, c, \partial_t c, r, \Omega)$.

Using the following properties of the Elliptic Projection: $\|\theta_c(0)\| \leq C^{st}h^r\|c_0\|$

$\|\partial_x\theta_c(0)\| \leq C^{st}h^{r-1}\|c_0\|$

$\|\eta_c(t)\| \leq C^{st}h^r\|c_0\|+C^{st}h^r\int_0^t\|\partial_t c(s)\|ds$

$\|\partial_t\eta_c(t)\| \leq C^{st}h^r\|\partial_t c(t)\|$

and the inequality (29) we obtain the approximation properties for the stability error θ_c. Recalling that $E_c(t) = \eta_c(t) + \theta_c(t)$ and $\|(c - c_h)(t)\| \leq \|\eta_c(t)\| + \|\theta_c(t)\|$ and the approximation properties (25), (26) and (29) for $\eta_c(t)$ and $\theta_c(t)$ we get:

$$\|(c - c_h)(t)\| \leq C^{st} h^r \|c(t)\| + \|\theta_c(0)\|$$

$$+C_1 C^{st} h^r \int_0^{t^*} (\|\partial_t c(t)\| + \|c(t)\| dt + C_1 \int_0^{t^*} \|(\rho - \rho_h)(t)\| dt$$

together with the estimate

$$\|\theta_c(0)\| = \|\Pi_h c(0) - c_h(0)\| \leq \|\Pi_h c_0 - c_0\| + \|c_0 - c_h(0)\| \leq C^{st} h^r \|c_0\| + \|c_0 - c_h(0)\|$$

Therefore finally we have:

$$\|(c - c_h)(t)\| \leq \|c_0 - c_h(0)\| + C^{st} h^r (\|c_0\| + \|c(t)\|)$$

$$+C_1 C^{st} h^r \int_0^{t^*} (\|\partial_t c(t)\| + \|c(t)\|) dt + C_1 \int_0^{t^*} \|(\rho - \rho_h)(t)\| dt \quad (29)$$

where the left hand side is bounded in terms of initial data.

We consider now the problem of deriving error estimates for the density's error (22).
From the weak formulation and the finite element formulation of the density and using the definition of the Elliptic Projection, we have that for all $\varphi \in V_h$

$$\langle \partial_t \theta_\rho, \varphi \rangle + \mu \langle \partial_x \theta_\rho, \partial_x \varphi \rangle$$

$$= -\langle \partial_t \eta_\rho, \varphi \rangle + \chi \langle \rho \partial_x c, \partial_\varphi \rangle - \chi \langle \rho_h \partial_x c_h, \partial_x \varphi \rangle$$

$$= -\langle \partial_t \eta_\rho, \varphi \rangle + \chi \langle \rho \partial_x c - \rho_h \partial_x c_h, \partial_x \varphi \rangle$$

We take $\varphi = \theta_\rho \in V_h$ to get

$$\frac{1}{2} \frac{d}{dt} \|\theta_\rho\|^2 + \mu \|\partial_x \theta_\rho\|^2$$

$$= -\langle \partial_t \eta_\rho, \theta_\rho \rangle + \chi \langle \rho \partial_x c, \partial_x \theta_\rho \rangle - \chi \langle \rho_h \partial_x c_h, \partial_x \theta_\rho \rangle$$

$$\leq (\|\partial_t \eta_\rho\| + \chi \langle \rho \partial_x c - \rho_h \partial_x c_h, \partial_x \theta_\rho \rangle$$

Then similar arguments using for θ_c let us to conclude.

5 Discretisation in time: Euler Implicit

In this section we consider the discretisation with respect to time which lead us to implement a numerical method to compute the solution of the problem.

5.1 Simplified model

In a first attempt we study a simplified version of our model considering our function $\chi = 1$. We rewrite the system (1) using the Euler Implicit scheme:

$$\begin{cases} \eta\rho^{n+1} - \mu\partial_{xx}\rho^{n+1} + \partial_x(\rho^{n+1}\partial_x c^{n+1}) = \eta\rho^n + f^{n+1} \\ \epsilon\eta c^{n+1} - \nu\partial_{xx}c^{n+1} + \beta c^{n+1} - \alpha\rho^{n+1} = \eta\epsilon c^n, \end{cases} \tag{30}$$

where $\eta = 1/\Delta t$ and the function f denotes the right hand side.
Now we consider the weak formulation of the problem taking $\varphi \in H^1$

$$\begin{cases} \eta\langle\rho^{n+1},\varphi\rangle + \mu\langle\partial_x\rho^{n+1},\partial_x\varphi\rangle - \langle\rho^{n+1}\partial_x c^{n+1},\partial_x\varphi\rangle = \langle\eta\rho^n + f^{n+1},\varphi\rangle \\ (\epsilon\eta+\beta)\langle c^{n+1},\varphi\rangle + \nu\langle\partial_x c^{n+1},\partial_x\varphi\rangle - \alpha\langle\rho^{n+1},\varphi\rangle = \langle\eta\epsilon c^n,\varphi\rangle \end{cases} \tag{31}$$

and then we substitute in it $\rho = \sum_{j=1}^N \rho_j\varphi_j$ and $c = \sum_{j=1}^N c_j\varphi_j$ to obtain:

$$\begin{cases} \eta\sum_{j=1}^N \rho_j^{n+1}\langle\varphi_j,\varphi_i\rangle + \mu\sum_{j=1}^N \rho_j^{n+1}\langle\varphi_j',\varphi_i'\rangle - \langle\sum_{k=1}^N \rho_k^{n+1}\varphi_k\sum_{j=1}^N c_j^{n+1}\varphi_j',\varphi_i'\rangle \\ = \eta\sum_{j=1}^N \rho_j^n\langle\varphi_j,\varphi_i\rangle + \langle f^{n+1},\varphi_i\rangle \\ (\epsilon\eta+\beta)\sum_{j=1}^N c_j^{n+1}\langle\varphi_j,\varphi_i\rangle + \nu\sum_{j=1}^N c_j^{n+1}\langle\varphi_j',\varphi_i'\rangle - \alpha\sum_{j=1}^N \rho_j^{n+1}\langle\varphi_j,\varphi_i\rangle = \eta\epsilon\sum_{j=1}^N c^n\langle\varphi_j,\varphi_i\rangle \end{cases} \tag{32}$$

Introducing the Stiffness matrix $K = \langle\varphi_j',\varphi_i'\rangle$, the Mass matrix $M = \langle\varphi_j,\varphi_i\rangle$ and the Triple matrix $T = \langle\varphi_k\varphi_j',\varphi_i'\rangle$ and considering the vector $u = [\rho_1,...,\rho_N,c_1,....,c_N]$ we have

$$\begin{cases} \eta\sum_{j=1}^N M_{ij}u_j^{n+1} + \mu\sum_{j=1}^N K_{ij}u_j^{n+1} - \sum_{k=1}^N\sum_{j=1}^N c_j^{n+1}T_{ijk}u_k^{n+1}u_{j+N}^{n+1} \\ = \eta\sum_{j=1}^N M_{ij}u_j^n + b_i^n \\ (\epsilon\eta+\beta)\sum_{j=1}^N M_{ij}u_{j+N}^{n+1} + \nu\sum_{j=1}^N K_{ij}u_{j+N}^{n+1} - \alpha\sum_{j=1}^N M_{ij}u_j^{n+1} = \eta\epsilon\sum_{j=1}^N M_{ij}u_{j+N}^n \end{cases} \tag{33}$$

where $b_i^n = \langle f^{n+1},\varphi_i\rangle$. To deal with the non linear term we have two possibility either linearising around u_k or linearising around u_{j+N}.
FIRST CASE: linearization around u_k
Using a matricial form we can rewrite our system in the form $AU = B$, where A is a $2N \times 2N$ matrix and U and B are two $1 \times 2N$ vectors:

$$\mathcal{A} = \begin{pmatrix} \eta M + \mu K & \tilde{T} \\ -\alpha M & (\epsilon\eta+\beta)M + \nu K) \end{pmatrix}$$

with $\tilde{T} = -\sum_{k=1}^{N} T_{ijk} u_k^n$.

$$\mathcal{U} = \begin{bmatrix} u_1^{n+1} \\ \vdots \\ u_{2N}^{n+1} \end{bmatrix} \quad \mathcal{B} = \begin{bmatrix} \eta M u_1^n + b_i^n \\ \vdots \\ \eta M u_n^n + b_i^n \\ \eta \epsilon M u_{n+1}^n \\ \vdots \\ \eta \epsilon M u_{2N}^n \end{bmatrix}$$

SECOND CASE: linearization around u_{j+N}

We use the same form of before and the only thing to change is the matrix A which is in this case

$$\mathcal{A} = \begin{pmatrix} \eta M + \mu K + \tilde{T} & 0 \\ -\alpha M & (\epsilon\eta + \beta)M + \nu K) \end{pmatrix}$$

with $\tilde{T} = -\sum_{j=1}^{N} T_{ijk} u_{j+N}^n$.

6 Numerical results and conclusions perspectives

To have some numerical results we implement a scilab program constructed to solve the Keller-Segel model for chemotaxis. Of course to realize this program we use all the previous considerations.
The algorithm consists mainly in the following steps:
-we define our initial data and the constant of the problem;
-we compute an exact solution for the problem and the respective right hand side;
-we define the mesh and we compute the matrix related to our system;
-we implement the Implicit Euler scheme in time to find the solution.
Using this algorithm first we can show the convergence of the constant solutions in the following results

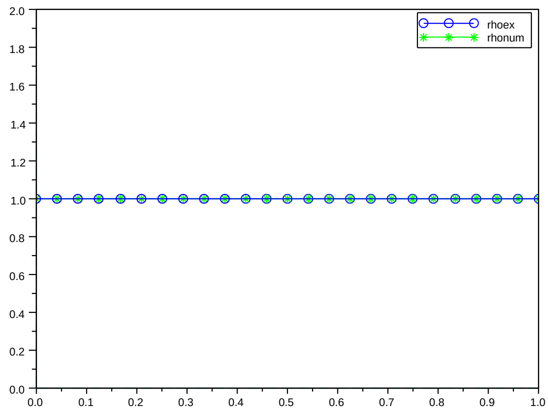

Figure 1: rho

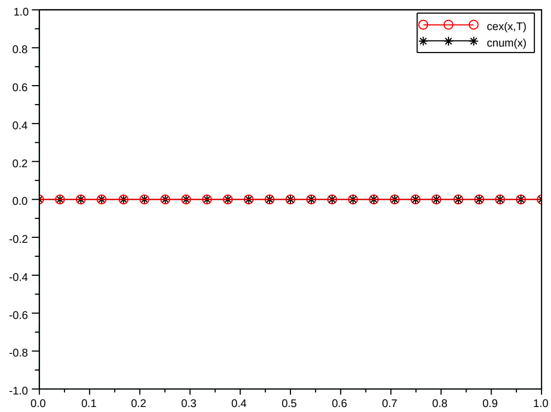

Figure 2: c

We can see in these pictures that there exists a perfect convergence for the constant solutions and these results are valid for every choice of the time step and the space step.
Then we do the same for non constant solutions and also in this case we obtain a good convergence between the exact and the numerical solutions, which can become perfect for particular values of our parameters. According to how we deal with the non linear term we consider separately the two different case.
First case: linearisation around u_k
We start our analysis considering the plot for the following value of the parameters:
-time step $dt = 4 * 10^{-4}$
-mesh $N = 35$
-$\epsilon, \mu, \nu = 1$ and $\alpha, \beta, \gamma = 0$
and we obtain:

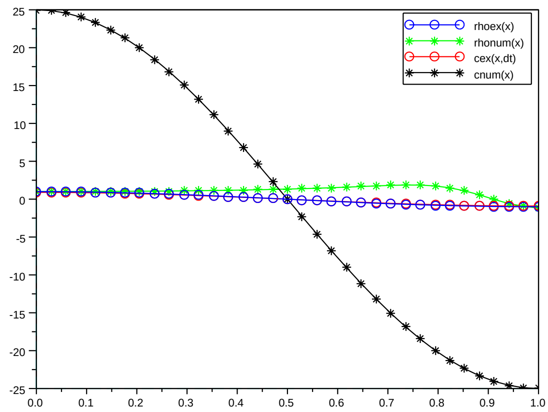

Figure 3: case1

As we can see the convergence is not very good in particular for $c(x,t)$. So first we try to change the value of the time step and we put $dt = 4 * 10^{-5}$ to obtain:

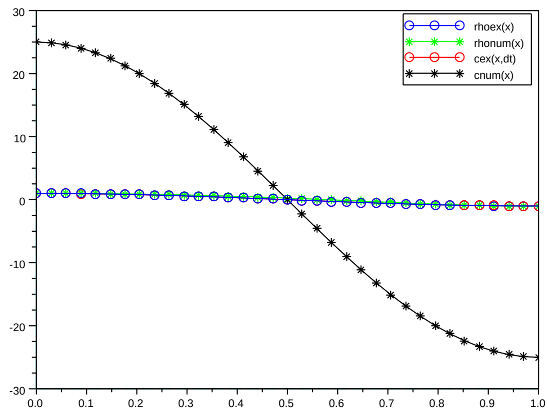

Figure 4: case2

but in fact with this change we see only an improvement on the convergence for ρ but not on c. So we try to change the other parameters N, N_t but as we can see the plot doesn't change a lot:

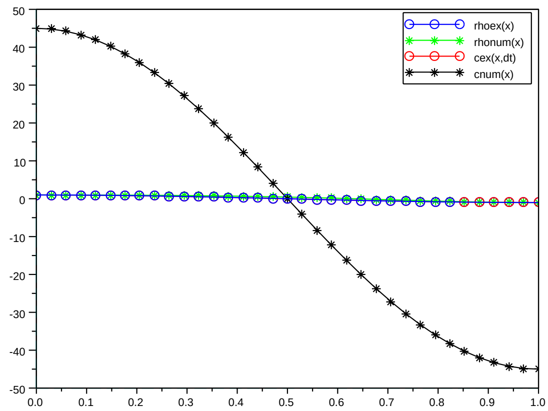

Figure 5: case3

We obtain the same results considering other values for the constants of our problem

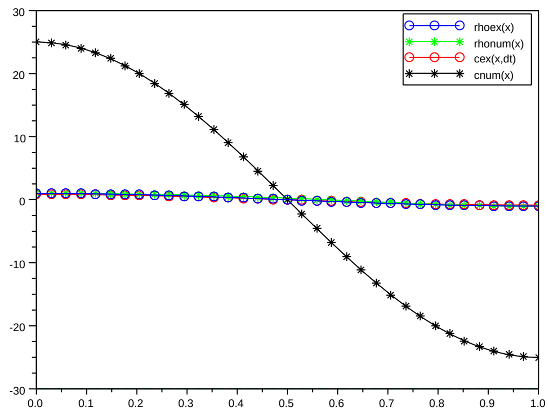

Figure 6: case4

Since in all this plot what we can see is that for the convergence of ρ there are no problem but for c the result is not so good in fact we have that the amplitude of the exact solution for c is too big with respect the amplitude of the numerical one, we think to reduce the amplitude of the exact solutions, putting a constant $a \leq 1$ in its expression.
So we start to choose $a = 0.5$ and we obtain:

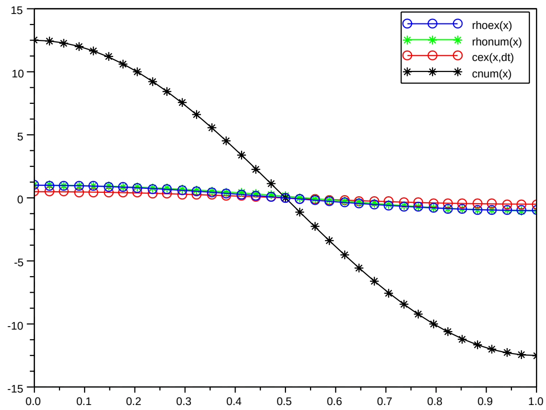

Figure 7: case5

Since also here we don't see any improvement we continue to reduce the value of a, so we put $a = 0.05$ in figure related to $case55$, then $a = 0.005$ in $case6$ and at the end $a = 0.0005$ for which we obtain the exact convergence in $case7$.

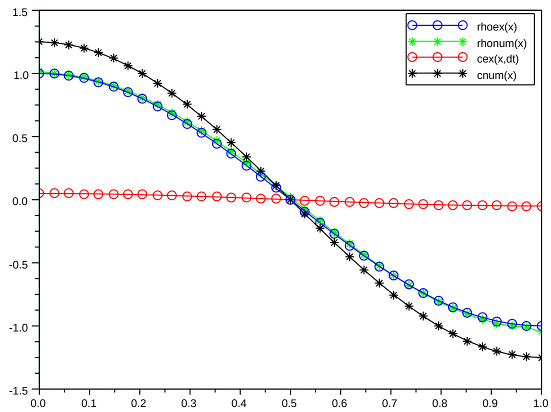

Figure 8: case55

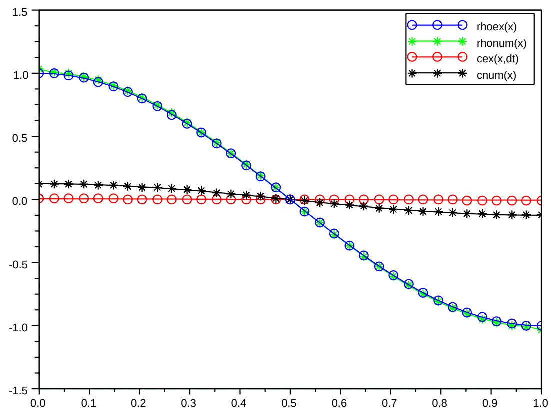

Figure 9: case6

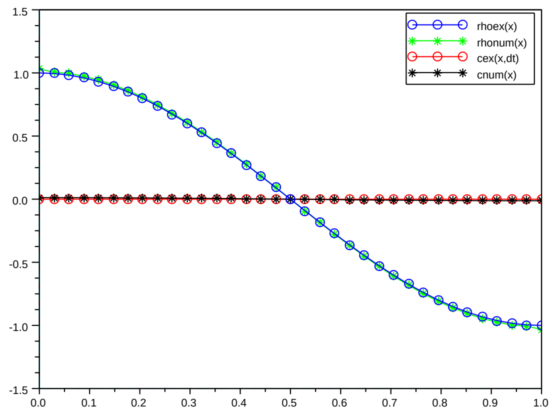

Figure 10: case7

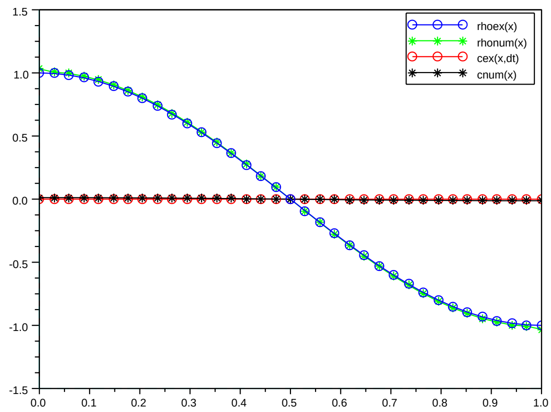

Figure 11: case8

We follow exactly the same procedure in the second case with linearisation around u_{j+N} and as we can see in the last picture we obtain the same convergence.

This work is manly an analytical study of a particular case of the Keller-Seghel model for chemotaxis. After a huge documentation on the different model for chemotaxis we chose to focus our interest in a one dimensional case, that allows us to use some particular properties of the Sobolev spaces, and we consider from the beginning some particular hypotheses on the different parameters present in the equation. Under our assumptions we prove the estimate of the L^2 energy and of the error between the analytical and the numerical solution and we show the numerical convergency of constant and non constant solutions for different values of the parameters. Naturally what we did in this paper can be extended in more general case, removing some of our assumption and considering other hypotheses. We limited our study in the one dimensional case but with similar arguments we can also study the behaviour of the system in higher dimensions.

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