Training. Relaxation finite element schemes for the incompressible Navier-Stokes equations

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Outline

1. Introduction
   - Navier-Stokes
   - Relaxation Finite Element Schemes for The Incompressible Navier-Stokes equations.
   - Implementation and Software.
   - Methodology
2. Stokes Equations
3. Steady State Navier-Stokes Equations
4. Incompressible Navier-Stokes Equations
5. Navier-Stokes equations with relaxation

Incompressible Navier-Stokes Equations

$$\begin{cases} \partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p & = f(x,t) \text{ on } \Omega \times [0,T] \\ \operatorname{div} u & = 0 \\ u|_{\partial\Omega} & = g(x,t) \text{ on } \partial\Omega \times [0,T] \end{cases} \tag{1}$$

- $\mathbb{R}^2 \supset \Omega$ space domain
- $[0, T] \subset \mathbb{R}$ time domain
- $u : \Omega \times [0, T] \longrightarrow \mathbb{R}^2$, $(x, t) \mapsto u(x, t)$ velocity
- $p : \Omega \times [0, T] \longrightarrow \mathbb{R}$, $(x, t) \mapsto p(x, t)$ pressure
- $\operatorname{div} u = 0$ describes incompressibility of the fluid.
- Given: Ω, T, f, g, ν
- Find: velocity u and pressure p

Navier-Stokes Equations with Relaxation.

$$\begin{cases} \partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p + \varepsilon \partial_{tt} u + \varepsilon \partial_t \nabla p & = f(x,t) \\ \mathrm{div} u & = 0 \\ u|_{\partial\Omega} & = g(x,t) \end{cases} \quad (2)$$

- $u : \Omega \times [0, T] \longrightarrow \mathbb{R}^2$, $(x, t) \mapsto u(x, t)$ velocity
- $p : \Omega \times [0, T] \longrightarrow \mathbb{R}$, $(x, t) \mapsto p(x, t)$ pressure
- Given: Ω, T, f, g, ν
- Find: u and p.

Advantage

Better description of the physics

Implementation and Software

Open Source and Free Software:

- Numerics: FreeFem++.
- Visualization: FreeFem++, gnuplot,R, (OpenDX)
- IDE: *FreeFem++-cs, Eclipse.*
- Publishing: LaTeX, LyX, latex-beamer, Inkscape,TextText.
- Version Control System: subversion.
- Programming Languages: FreeFem++, python, C++, R.
- Programming paradigm: Literate Programming

Methodology

- Fix the goal. Get Familar with the problem.
- Collect Information. Find the starting point.
- Go step by step increasing the complexity of the problem.

$$\begin{cases} \partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p + \varepsilon \partial_{tt} u + \varepsilon \partial_t \nabla p & = f(x,t) \\ \mathrm{div} u & = 0 \\ u|_{\partial\Omega} & = g(x,t) \end{cases}$$

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Stokes equations

$$\begin{cases} -\nu\Delta\tilde{u} + \nabla p & = \tilde{f}(x,t) \\ \text{div}\tilde{u} & = 0 \\ \tilde{u}|_{\partial\Omega} & = \tilde{g}(x,t) \end{cases}$$

Bring the equation in tho the form

$$\begin{cases} -\nu\Delta u + \nabla p & = f(x,t) \\ \text{div}u & = 0 \\ u|_{\partial\Omega} & = 0 \end{cases}$$

In order to solve this equation we need to introduce some new concepts

- Mixed Finite elements for variational formulation
- Inf-sup conditions

Mixed Finite elements, Variational Formulation

$$\begin{cases} -\nu\Delta u + \nabla p &= f(x,t) \qquad \mid \cdot\, \phi \in X,\ \int_\Omega \\ \text{div} u &= 0 \qquad \mid \cdot\, \psi \in M,\ \int_\Omega \\ u|_{\partial\Omega} &= 0 \end{cases}$$

- Space for Velocity u: $X = H^1(\Omega)^2$
- Space for pressure $M = \{q \in L^2(\Omega) : \int_\Omega q dx = 0\}$

$$\begin{cases} a(u,\phi) + b(\phi,p) &= F(\phi) \qquad \forall \phi \in X \\ b(u,\psi) &= 0 \qquad \forall \psi \in M \end{cases}$$

with

$$a(v,\phi) = \nu < \nabla v, \nabla\phi >, \qquad b(\phi,\psi) = - < \text{div}\phi, \psi >$$

Inf-Sup, Existence and Uniqueness of the Solution

$$\begin{cases} a(u,\phi)+b(\phi,p) & = F(\phi) \qquad \forall \phi \in X \\ b(u,\psi) & = 0 \qquad \forall \psi \in M \end{cases}$$

Solve equations for $u \in Z = \{\phi \in X | b(\phi,\psi) = 0 \quad \forall \psi \in M\}$, that means solve

$$a(u,\phi) + \cancel{b(\phi,p)} = F(\phi) \qquad \forall \phi \in Z$$

Solution exists due to coercivity and continuity of $a(u,\phi)$
Use u to solve

$$b(\phi,p) = F(\phi) - a(u,\phi) \qquad \forall \phi \in X$$

Problem: ϕ and p are from different spaces:
Solution: use inf-sup-Condition $\sup_{v \in X} \frac{b(\phi,\psi)}{||\phi||_H} \geq \alpha ||\psi||_M \quad \forall v \in X$ instead of coercivity

Relationship between Inf-Sup and Coercivity

Let $c(u,v)$ be an coercive bilinear form in a Hilbert space V, scalar product $<>_H$ and appropriate norm $||\cdot||_V$. than we have

$$\begin{aligned} c(u,v) &\geq \alpha||u||_H||v||_H \quad \forall u,v \in H \\ \Longrightarrow \frac{c(u,v)}{||v||_H} &\geq \alpha||u||_H \quad \forall u,v \in H \\ \Longrightarrow \sup_{v\in H} \frac{c(u,v)}{||v||_H} &\geq \alpha||u||_H \quad \forall u \end{aligned}$$

For existence and uniqueness of $b(\phi,p) = F(\phi) - a(u,\phi) \forall\phi \in X$

$$\Longrightarrow \sup_{v\in X} \frac{b(\phi,\psi)}{||\phi||_H} \geq \alpha||\psi||_M \quad \forall v \in X$$

is enough.

Stokes Equations, Summary

Stokes equations	Poisson's equations
$\begin{cases} -\nu\Delta u + \nabla p &= f(x,t) \\ \operatorname{div} u &= 0 \\ u\|_{\partial\Omega} &= 0 \end{cases}$	$\begin{cases} -\nu\Delta u &= f(x,t) \\ u\|_{\partial\Omega} &= 0 \end{cases}$
Mixed FEM:	"not mixed" FEM
$\begin{cases} a(u,\phi) + b(\phi,p) &= F(\phi),\ \forall\phi\in X \\ b(u,\psi) &= 0,\ \forall\psi\in M \end{cases}$	$a(u,\phi) + = F(\phi),\ \forall\phi\in X$
Inf-Sup	Coercivity
$\sup_{v\in X}\frac{b(\phi,\psi)}{\|\|\phi\|\|_H} \geq \alpha\|\|\psi\|\|_M,\ \forall v\in X$	$a(u,v)\geq\alpha\|\|u\|\|_X\|\|v\|\|_X$

Stokes Equations, Remarks to Numerics

$$\begin{cases} -\nu\Delta u + \nabla p &= f(x,t) \\ \text{div} u &= 0 \\ u|_{\partial\Omega} &= 0 \end{cases}$$

- $\text{div} u = 0$. X_h is not a subspace of the Solution!
  - Source of instabilities.
  - Construction of divergence preserving FEM-spaces of high order of precision is not simple
- Possible solutions:
  - Relaxation of the divergence free condition $\text{div} u \approx \text{div} u + \varepsilon u$
  - Special spaces like e.g. Crouzeix-Raviart for lower order precision

Stokes Equations simulations, Construction of Exact Solution

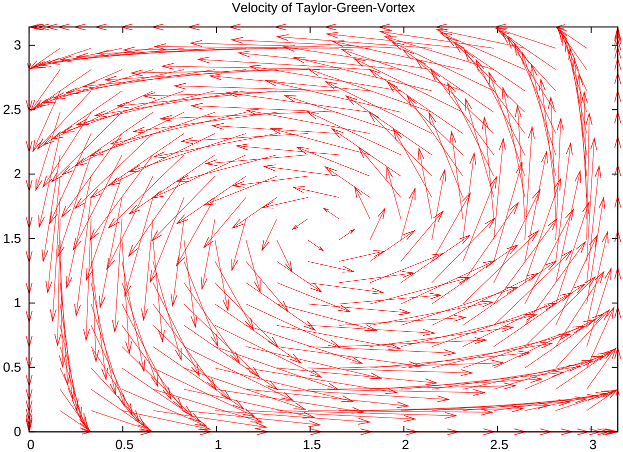
Taylor-Green Vortex

$$\begin{cases} \partial_t w + w \cdot \nabla w - \nu \Delta w + \nabla p & = 0 \\ \text{div} w & = 0 \\ w\big|_{\partial\Omega} & = w\big|_{\partial\Omega} \end{cases}$$

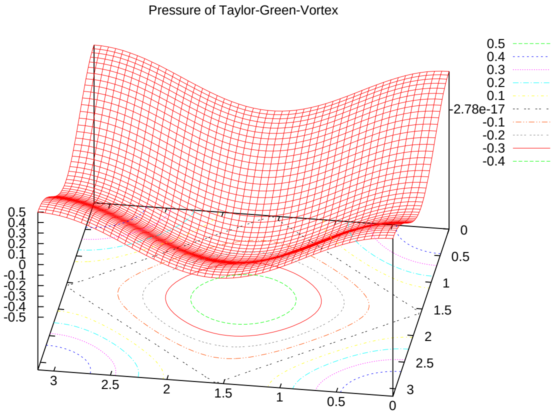
Stokes equations

$$\begin{cases} -\nu \Delta w + \nabla p & = \underbrace{-(\partial_t w + w \cdot \nabla w)}_{f(x)}\big|_{t=c} \\ \text{div} w & = 0 \\ w\big|_{\partial\Omega} & = w\big|_{\partial\Omega, t=c} \end{cases}$$

Taylor-Green Vortex, Velocity



Taylor-Green Vortex, Pressure



Stokes Equations, Simulations

Mixed FEM: Taylor-Hood

$X_h = \{\phi \in H^1([0,\pi])^2 : \phi|_T \in P^2 \forall T \in \mathcal{T}_h\}$

$M_h = \{\psi \in H^1([0,\pi]) : \psi|_T \in P^1 \forall T \in \mathcal{T}_h\}$

Mesh: $N \times N$ uniform mesh

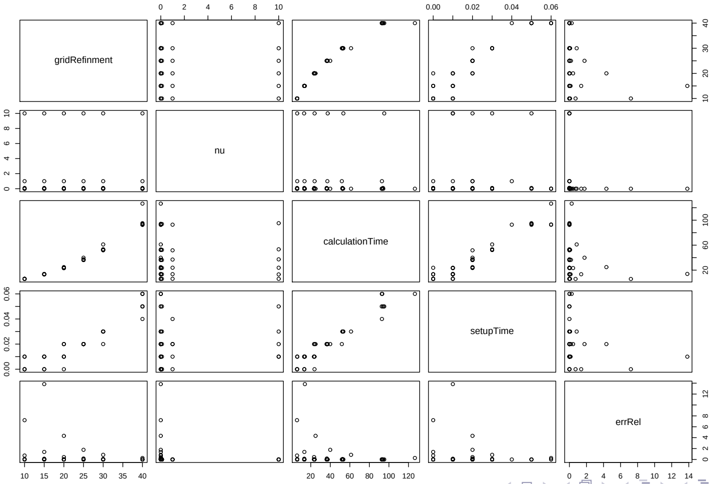
Parameters:

ν	10	1.0	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	0

N	10	15	20	25	30	40

Analytical reference: Taylor-Green Vortex

Stokes Equations, Postprocessing of Simulations

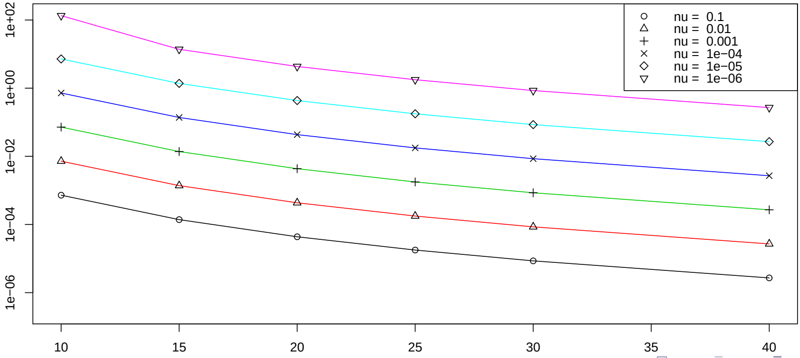


Stokes Equations, Postprocessing

Influence of grid refinement on relative error, Taylor-Hood

$X_h = \{\phi \in H^1([0,\pi])^2 : \phi|_T \in P^2 \forall T \in T_h\}$,

$M_h = \{\psi \in H^1([0,\pi]) : \psi|_T \in P^1 \forall T \in T_h\}$



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Steady State Navier-Stokes Equations

$$\begin{cases} u \cdot \nabla u - \nu \Delta u + \nabla p & = f(x,t) \\ \mathrm{div} u & = 0 \\ u|_{\partial\Omega} & = g(x,t) \end{cases}$$

- Problem: Nonlinear part $u \cdot \nabla u$.
- Possible Solution: Iterative method
  $u^n \cdot \nabla u^{n+1} - \nu \Delta u^{n+1} + \nabla p^{n+1} = f(x,t)$ (Oseen Problem)
  - Required: not too small ν, small f
- Possible reference, exact solution: Taylor-Green Vortex w

$$\begin{cases} w \cdot \nabla w - \nu \Delta w + \nabla p & = \underbrace{\partial_t w}_{f(x)} |_{t=c} \\ \mathrm{div} w & = 0 \\ w|_{\partial\Omega} & = w|_{\partial\Omega, t=c} \end{cases}$$

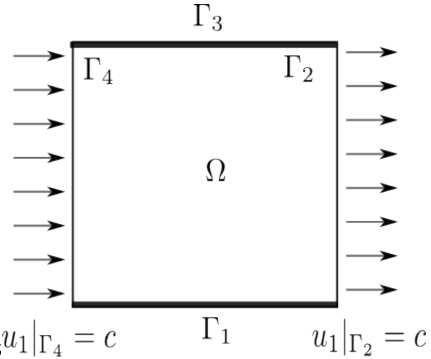
Steady State Navier-Stokes equations, Simulations

Mixed FEM: Taylor-Hood

$X_h = \{\phi \in H^1([0,\pi])^2 : \phi|_T \in P^2 \forall T \in T_h\}$

$M_h = \{\psi \in H^1([0,\pi]) : \psi|_T \in P^1 \forall T \in T_h\}$

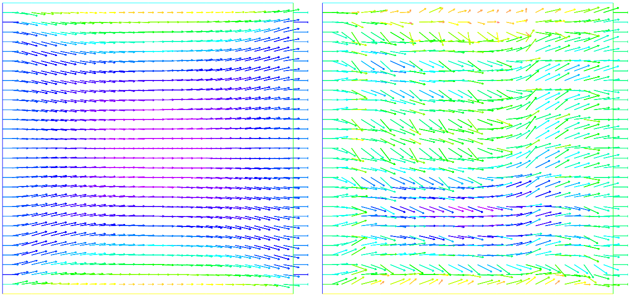
Mesh: $N \times N$ uniform mesh



Boundary conditions

Steady State Navier-Stokes equations.

Figure:



Simulation with $k = 20$, grid=10x10: $\nu = 0.1$ left, $\nu = 10^{-7}$right, $X_h = P2$, $M_h = P1$

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$$\begin{cases} \partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p & = f(x,t) \\ \mathrm{div} u & = 0 \\ u|_{\partial\Omega} & = g(x,t) \end{cases}$$

- Problem: Nonlinear part $u \cdot \nabla u$ and time derivative:
- Possible Solution: Discretization in time and linearisation (FreeFem++ *convex* operator)

$$\underbrace{\partial_t u + u^n \cdot \nabla u^n}_{\frac{1}{\Delta t}(u^{n+1} - u^n \circ X^n)} - \nu \Delta u^{n+1} + \nabla p^{n+1} = f^{n+1}$$

- Possible reference: Taylor-Green Vortex

$$\begin{cases} \partial_t w + w \cdot \nabla w - \nu \Delta w + \nabla p & = 0 \\ \mathrm{div} w = 0, & = 0 \end{cases}$$

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Navier Stokes equations with Relaxation

$$\begin{cases} \partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p & + \varepsilon \partial_{tt} u + \varepsilon \partial_t \nabla p = f(x,t) \\ divu & = 0 \\ u|_{\partial\Omega} & = g(x,t) \qquad on\ \partial\Omega \times [0,T] \end{cases}$$

- Problems: Time discretization $\varepsilon \partial_{tt} u + \varepsilon \partial_t \nabla p$
- Stabilities and Instabilities caused by ε parameters
- Possible solution: FD for time discretization special divergence preserving FEM spaces e.g. Crouzeix-Raviart
- Possible reference: Taylor-Green Vortex

$$\begin{cases} \partial_t w + w \cdot \nabla w - \nu \Delta w + \nabla p + \varepsilon \partial_{tt} w + \varepsilon \partial_t \nabla q & = \underbrace{\varepsilon \partial_{tt} w + \varepsilon \partial_t \nabla q}_{f(x,t)} \\ \operatorname{div} w = 0, & = 0 \end{cases}$$

Variational Formulation

$$
\begin{aligned}
0 \approx\ & \frac{1}{\Delta t} < u^{n+1}, \phi > + \frac{\varepsilon}{\Delta t^2} < u^{n+1}, \phi > + \nu < \nabla u^{n+1}, \nabla \phi > \\
& - < \mathrm{div}\phi, p^{n+1} > + \frac{\varepsilon}{\Delta t} < \mathrm{div}\phi, p^{n+1} > \\
& - \frac{1}{\Delta t} < \mathrm{convect}(u^n, -\Delta t), \phi > \\
& + \frac{\varepsilon}{\Delta t^2} < -2u^n + u^{n-1}, \phi > - \frac{\varepsilon}{\Delta t} < \mathrm{div}\phi, p^{n+1} > \\
& + \int_{\partial\Omega} \frac{\partial}{\partial n} u^{n+1} \phi ds + \int_{\partial\Omega} (\phi \cdot n) p ds \\
& + \frac{\varepsilon}{\Delta t^2} \int_{\partial\Omega} \frac{\partial}{\partial n} (-2u^n + u^{n-1}) \phi ds - \frac{\varepsilon}{\Delta t} \int_{\partial\Omega} (\phi \cdot n) p^n ds
\end{aligned}
$$