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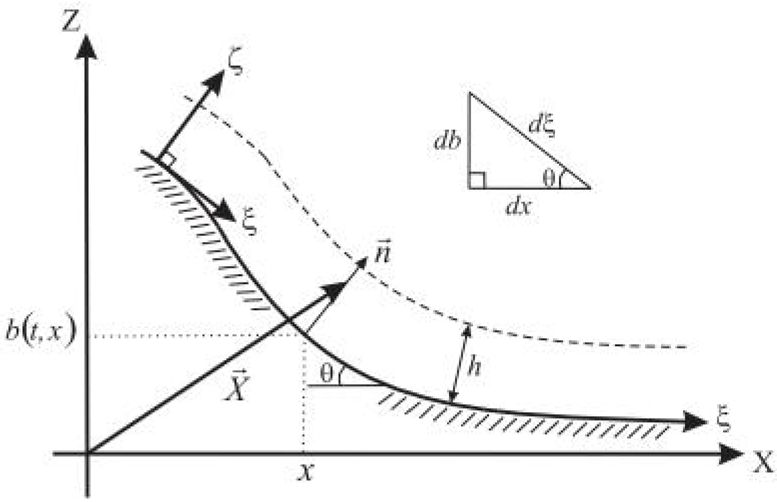
Shallow watter equation on the basal surface:



$$\frac{\partial h}{\partial \tau} + \frac{\partial}{\partial \xi}(hq_\xi) = -\mathcal{M} \; ,$$

$$\frac{\partial (hu)}{\partial \tau} + \frac{\partial}{\partial \xi}\left(huq_\xi + \frac{\beta_\xi h^2}{2}\right) = -\mathcal{M}u + hs_\xi \; ,$$

$$\frac{\partial}{\partial t}w + \frac{\partial}{\partial x}f_x(w) + \frac{\partial}{\partial y}f_y(w) = 0$$

$$w = \begin{bmatrix} h \\ hu \\ hv \end{bmatrix} f_x(w) = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ hvu \end{bmatrix} f_y(w) = \begin{bmatrix} hv \\ huv \\ hv^2 + \frac{1}{2}gh^2 \end{bmatrix}$$

or in 1-D case:

$$\frac{\partial}{\partial t}w + \frac{\partial}{\partial x}f_x(w) = 0$$

$$w = \begin{bmatrix} h \\ hu \end{bmatrix} f_x(w) = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{bmatrix}$$

so it can be written as:

$$\begin{gathered} \frac{\partial}{\partial t}h + \frac{\partial}{\partial x}(hu) = 0 \\ \frac{\partial}{\partial t}(hu) + \frac{\partial}{\partial x}(hu^2 + \frac{1}{2}gh^2) = 0 \end{gathered}$$

Mathematical properties of 1-D equation.

Jacobian of $f_x(h, hu)$.

$$A = \frac{\partial f_x}{\partial w} = \begin{pmatrix} 0 & 1 \\ -u^2 + gh & 2u \end{pmatrix}$$

The matrix A have real eigenvalues and linearly independent eigenvectors.

$$\lambda_1 = u + \sqrt{gh} \qquad \lambda_2 = u - \sqrt{gh}.$$

Numerical approximation (Finite volumes).

We integrate equation on the volume $\left(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right)$ and the time inteval $\left(t^n, t^{n+1}\right)$:

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (w(t^{n+1}, x) - w(t^n, x))dx$$

$$+ \int_{t^n}^{t^{n+1}} (f_x(w(t, x_{i+\frac{1}{2}})) - f_x(w(t, x_{i-\frac{1}{2}})))dt = 0$$

$$(t^{n+1} - t^n)(\phi_{i+\frac{1}{2}} - \phi_{i-\frac{1}{2}}) + (x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}})(\bar{w}_i^{n+1} - \bar{w}_i^n) = 0$$

where:

$$\bar{w}_i^n = \frac{1}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (w(t^n, x)dx$$

$$\phi_{i+\frac{1}{2}}^n = \frac{1}{t^{n+1} - t^n} \int_{t^n}^{t^{n+1}} (f_x(w(t, x_{i+\frac{1}{2}}))dt$$

If we can approximate fluks $\phi^n_{i+\frac{1}{2}}$ such that $\phi^n_{i+\frac{1}{2}} \approx \phi(\bar{w}^*_i, \bar{w}^*_{i+1})$.

Then scheme becomes:

$$\bar{w}_i^{n+1} = \bar{w}_i^n + \frac{t^{n+1} - t^n}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}}[\phi(\bar{w}^*_i, \bar{w}^*_{i+1}) - \phi(\bar{w}^*_{i-1}, \bar{w}^*_i)]$$

Explicit scheme:

$$\bar{w}_i^{n+1} = \bar{w}_i^n + \frac{t^{n+1} - t^n}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}}[\phi(\bar{w}^n_i, \bar{w}^n_{i+1}) - \phi(\bar{w}^n_{i-1}, \bar{w}^n_i)]$$

Approximation of the flux (Rusanov scheme):

Scheme is stable if we approximate $\phi^n_{i+\frac{1}{2}} \approx \phi(\bar{w}^n_i, \bar{w}^n_{i+1})$ like:

$$\phi^n_{i+\frac{1}{2}} \approx \phi(\bar{w}^n_i, \bar{w}^n_{i+1}) = \frac{1}{2}(f_x(\bar{w}^n_i) + f_x(\bar{w}^n_{i+1}) + \mid \xi \mid (\bar{w}^n_i - \bar{w}^n_{i+1}))$$

where $\mid \xi \mid \geq max(\mid \lambda \mid)_{w \in \left\{ \bar{w}^n_i \bar{w}^n_{i+1} \right\}}$.

Initial and boundary conditions.

- **Initial conditions** - In time $t = 0$, h and u are known.

- **Boundary conditions** - When we are calculating $\bar{w}_i^{n+1}$ and we are on the boundary, $\bar{w}_0^n$ or $\bar{w}_{m+1}^n$ are unknown. Since we are solving equation in the closed box, that means that values $\bar{w}_0^n = \begin{pmatrix} h_0^n \\ hu_0^n \end{pmatrix} = \begin{pmatrix} h_1^n \\ -hu_1^n \end{pmatrix}$ and $\bar{w}_{m+1}^n = \begin{pmatrix} h_{m+1}^n \\ hu_{m+1}^n \end{pmatrix} = \begin{pmatrix} h_m^n \\ -hu_m^n \end{pmatrix}$.

Second order of approximation.

So far scheme was considering that on intervals $\left(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right)$, $i = 1, .., m$, functions h and hu are constant. Find aproximation:

- volumes on the intervals remain the same.

- functions on the intervals are linear.

- Total variation of the aproximated functions is smaller.

Minmod method.

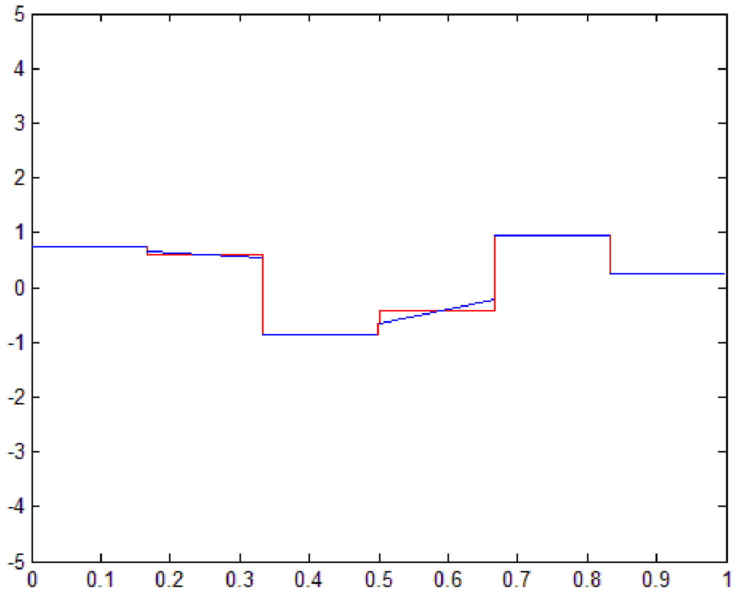
Max principle:

$$h_k(x_{i+\frac{1}{2}}) \in (h(x_i), h(x_{i+1})) \qquad and \qquad h_k(x_{i-\frac{1}{2}}) \in (h(x_{i-1}), h(x_i)).$$

- $h_2(x) = \frac{h(x_{i+1})-h(x_i)}{dx}(x - x_i) + h(x_i)$

- $h_3(x) = \frac{h(x_i)-h(x_{i-1})}{dx}(x - x_i) + h(x_i)$.

The function that has satisfied max principle can be approximation. If non of the $h_k(x)$ is good approximation then we can for that interval remain first order approximation.


5
4
3
2
1
0
-1
-2
-3
-4
-5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Second order approximation of the flux we obtain with:

$$\phi^n_{i+\frac{1}{2}} = \phi(\bar{w}^n_i, \bar{w}^n_{i+1}) = \frac{1}{2}(f_x((\bar{w}^n_i)_R) + f_x((\bar{w}^n_{i+1})_L)$$

$$+ \mid \xi \mid ((\bar{w}^n_i)_R - (\bar{w}^n_{i+1})_L))$$

where $(\bar{w}^n_i)_R$ marks value of the corresponding approximated function in the right point of the interval i. And $(\bar{w}^n_{i+1})_L$ marks value of the corresponding approximated function in the left point of the interval $i+1$.

Second order approximation scheme - for this scheme we are using predictor and corector.

$$\bar{w}_i^* = \bar{w}_i^n + \frac{\frac{1}{2}(t^{n+1} - t^n)}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}}[\phi(\bar{w}_i^n, \bar{w}_{i+1}^n) - \phi(\bar{w}_{i-1}^n, \bar{w}_i^n)]$$

$$\bar{w}_i^{n+1} = \bar{w}_i^n + \frac{t^{n+1} - t^n}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}}[\phi(\bar{w}_i^*, \bar{w}_{i+1}^*) - \phi(\bar{w}_{i-1}^*, \bar{w}_i^*)]$$

Mathematical properties of 2-D equation.

$$A = \frac{\partial f_x}{\partial w} = \begin{pmatrix} 0 & 1 & 0 \\ -u^2 + gh & 2u & 0 \\ -uv & v & u \end{pmatrix}$$

$$B = \frac{\partial f_y}{\partial w} = \begin{pmatrix} 0 & 1 & 0 \\ -uv & u & v \\ -v^2 + gh & 2v & 0 \end{pmatrix}$$

The matrix A is easily shown to have real eigenvalues and linearly independent eigenvectors.

$$\lambda_1 = u + \sqrt{gh} \qquad \lambda_2 = u - \sqrt{gh} \qquad \lambda_3 = u.$$

and for B we have the same:

$$\lambda_1 = v + \sqrt{gh} \qquad \lambda_2 = v - \sqrt{gh} \qquad \lambda_3 = v.$$

Numerical approximation of 2-D equation.

We integrate equation on the volume $\left(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right) \times \left(y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}\right)$ and time interval $\left(t^n, t^{n+1}\right)$:

$$(t^{n+1} - t^n)h_y(\phi^n_{i+\frac{1}{2},j} - \phi^n_{i-\frac{1}{2},j}) + (t^{n+1} - t^n)h_x(\phi^n_{i,j+\frac{1}{2}} - \phi^n_{i,j-\frac{1}{2}})$$

$$+h_x h_y(\bar{w}_i^{n+1} - \bar{w}_i^n) = 0$$

$$\bar{w}_i^n = \frac{1}{h_x h_y} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (w(t^n, x, y) dx dy$$

$$\phi_{i+\frac{1}{2},j}^n = \frac{1}{(t^{n+1} - t^n) h_y} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \int_{t^n}^{t^{n+1}} (f_x(w(t, x_{i+\frac{1}{2}}, y)) dt dy$$

$$\phi_{i,j+\frac{1}{2}}^n = \frac{1}{(t^{n+1} - t^n) h_x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{t^n}^{t^{n+1}} (f_x(w(t, x, y_{j+\frac{1}{2}})) dt dx$$

If we can approximate fluks $\phi^n_{i,j+\frac{1}{2}}$ such that $\phi^n_{i,j+\frac{1}{2}} = \phi(\bar{w}^n_{i,j}, \bar{w}^n_{i,j+1})$, and $\phi^n_{i+\frac{1}{2},j} = \phi(\bar{w}^n_{i,j}, \bar{w}^n_{i+1,j})$ the we have obtained scheme.

Approximation of the flux:

Scheme is stable if we approximate:

$$\phi^n_{i+\frac{1}{2},j} = \phi(\bar{w}^n_{i,j}, \bar{w}^n_{i+1,j}) = \frac{1}{2}(f_x(\bar{w}^n_{i,j}) + f_x(\bar{w}^n_{i+1,j}) + \mid \xi_1 \mid (\bar{w}^n_{i,j} - \bar{w}^n_{i+1,j}))$$

and

$$\phi^n_{i,j+\frac{1}{2}} = \phi(\bar{w}^n_{i,j}, \bar{w}^n_{i,j+1}) = \frac{1}{2}(f_y(\bar{w}^n_{i,j}) + f_y(\bar{w}^n_{i,j+1}) + \mid \xi_2 \mid (\bar{w}^n_{i,j} - \bar{w}^n_{i,j+1}))$$

where $\mid \xi_1 \mid \geq max(\mid \lambda \mid_{f_x})_{w \in \left\{ \bar{w}^n_{i,j}, \bar{w}^n_{i+1,j} \right\}}$ and $\mid \xi_2 \mid \geq max(\mid \lambda \mid_{f_y})_{w \in \left\{ \bar{w}^n_{i,j}, w^n_{i,j+1} \right\}}$.