Numerical solution for shallow water equation on the basal surface

MathMods student: Aleksandar Mojsic

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Vector equation:

$$\frac{\partial}{\partial t}w + \frac{\partial}{\partial x}f_x(w) + \frac{\partial}{\partial y}f_y(w) = 0$$

where:

$$w = \begin{bmatrix} h \\ hu \\ hv \end{bmatrix} f_x(w) = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ hvu \end{bmatrix} f_y(w) = \begin{bmatrix} hv \\ huv \\ hv^2 + \frac{1}{2}gh^2 \end{bmatrix}$$

or in 1-D case:

$$\frac{\partial}{\partial t}w + \frac{\partial}{\partial x}f_x(w) = 10$$

where:

$$w = \begin{bmatrix} h \\ hu \end{bmatrix} f_x(w) = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{bmatrix}$$

so it can be written as:

$$\begin{array}{c} \frac{\partial}{\partial t}h + \frac{\partial}{\partial x}(hu) = 0 \\ \frac{\partial}{\partial t}(hu) + \frac{\partial}{\partial x}(hu^2 + \frac{1}{2}gh^2) = 0 \end{array}$$

Is shallow watter equation on the basal surface.

Lets first concentrate on the 1-problem.

Mathematical properties of 1-D equation.

If we observe function $f_x(h, hu)$ it is easy to calculate eigenvalues of the Jacobean of $f_x(h, hu)$.

$$A = \frac{\partial f_x}{\partial w} = \begin{pmatrix} 0 & 1 \\ -u^2 + gh & 2u \end{pmatrix}$$

The matrix A is easily shown to have real eigenvalues and linearly independent eigenvectors. In particular, the eigenvalues of A are given:

$$\lambda_1 = u + \sqrt{gh} \qquad\qquad \lambda_2 = u - \sqrt{gh}.$$

Numerical approximation(Finite volumes).

We discretaze domain with uniform mesh on the space domain and uniform mesh on the time domain. On the space domain we have m intervals: $\left(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right)$, $i = 1, .., m$.

We integrate equation on the interval $\left(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right) \times \left(t^n, t^{n+1}\right)$:

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{t^n}^{t^{n+1}} \frac{\partial}{\partial t} w dt dx + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{t^n}^{t^{n+1}} \frac{\partial}{\partial x} f_x(w) dt dx = 0$$

and get:

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (w(t^{n+1}, x) - w(t^n, x)) dx + \int_{t^n}^{t^{n+1}} (f_x(w(t, x_{i+\frac{1}{2}})) - f_x(w(t, x_{i-\frac{1}{2}}))) dt = 0$$

Shortly written:

$$(t^{n+1}-t^n)(\phi_{i+\frac{1}{2}}-\phi_{i-\frac{1}{2}})+(x_{i+\frac{1}{2}}-x_{i-\frac{1}{2}})(\bar{w}_i^{n+1}-\bar{w}_i^n)=0$$

where:

$$\bar{w}_i^n=\frac{1}{x_{i+\frac{1}{2}}-x_{i-\frac{1}{2}}}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}(w(t^n,x)dx$$

$$\phi_{i+\frac{1}{2}}^n=\frac{1}{t^{n+1}-t^n}\int_{t^n}^{t^{n+1}}(f_x(w(t,x_{i+\frac{1}{2}}))dt$$

If we can approximate flux $\phi_{i+\frac{1}{2}}^n$ such that $\phi_{i+\frac{1}{2}}^n \approx \phi(\bar{w}_i^n,\bar{w}_{i+1}^n)$, $\forall i$. Then scheme becomes:

$$\bar{w}_i^{n+1}=\bar{w}_i^n+\frac{t^{n+1}-t^n}{x_{i+\frac{1}{2}}-x_{i-\frac{1}{2}}}[\phi(\bar{w}_i^n,\bar{w}_{i+1}^n)-\phi(\bar{w}_{i-1}^n,\bar{w}_i^n)]$$

Approximation of the flux:

Scheme is stable if we approximate $\phi_{i+\frac{1}{2}}^n \approx \phi(\bar{w}_i^n,\bar{w}_{i+1}^n)$ like:

$$\phi_{i+\frac{1}{2}}^n \approx \phi(\bar{w}_i^n,\bar{w}_{i+1}^n)=\frac{1}{2}(f_x(\bar{w}_i^n)+f_x(\bar{w}_{i+1}^n)+\mid\xi\mid(\bar{w}_i^n-\bar{w}_{i+1}^n))$$

where $\mid\xi\mid\geq max(\mid\lambda\mid)_{w\in\{\bar{w}_i^n\bar{w}_{i+1}^n\}}$.

Now it is importante to say what are the initial and what are the boundary conditions.

Initial and boundary conditions.

In time $t = 0$, h and u are known. When we are calculating $\bar{w}_i^{n+1}$ and we are on the boundary, so $\bar{w}_0^n$ or $\bar{w}_{m+1}^n$ are unknown. Since we are solving equation in the closed box. That means that values in $\bar{w}_0^n = \begin{pmatrix} h_0^n \\ hu_0^n \end{pmatrix}$ and $\bar{w}_{m+1}^n = \begin{pmatrix} h_{m+1}^n \\ hu_{m+1}^n \end{pmatrix}$ must be fixed such that velocity on the boundary point is zero and h has the same value. That is satisfied if $\bar{w}_0^n = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \bar{w}_1^n$, ($\bar{w}_{m+1}^n = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \bar{w}_m^n$). It is important to notice that $\bar{w}_0^n$ and $\bar{w}_{m+1}^n$ are values that are not calculated in the domain. But since we are simulating closed box, above conditions are boundary conditions that must be satisfied.

Second order of approximation.

So far scheme was considering that on intervals $\left(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right)$, $i = 1, .., m$, functions h and hu are constant. Second order approximation we can obtain if we try to modify values of those functions, knowing their volumes on the intervals, such that:

- volumes on the intervals remain the same.
- functions on the intervals are linear.
- Total variation of the functions is smaller.

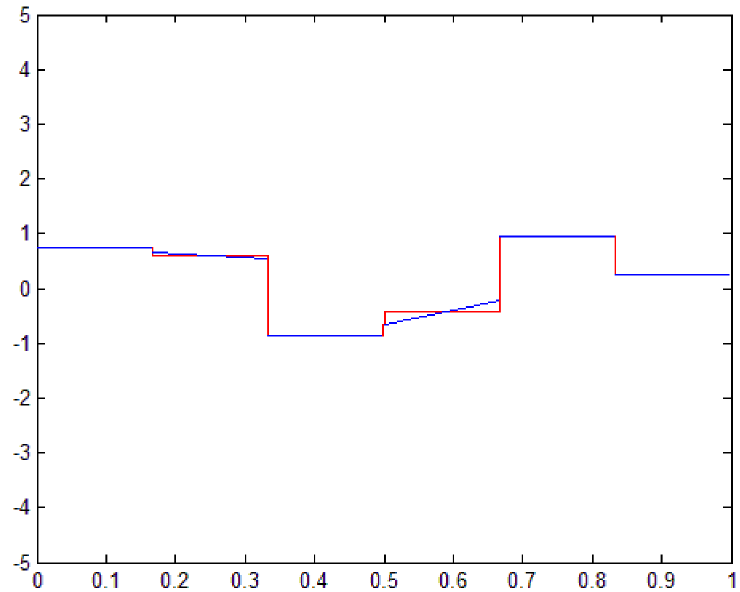
This problem can be solved by **max principle**.

For example in some time step, on the intervals $\left(x_{i-\frac{3}{2}}, x_{i-\frac{1}{2}}\right)$, $\left(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right)$, $\left(x_{i+\frac{1}{2}}, x_{i+\frac{3}{2}}\right)$, function h has constant values: $h(x_{i-1})$, $h(x_i)$ and $h(x_{i+1})$. If we approximate $h(x)$ on the interval $\left(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right)$ with: $h_1(x) = \frac{h(x_{i+1})-h(x_{i-1})}{2dx}(x-x_i)+h(x_i)$ or $h_2(x) = \frac{h(x_{i+1})-h(x_i)}{dx}(x-x_i)+h(x_i)$ or $h_3(x) = \frac{h(x_i)-h(x_{i-1})}{dx}(x-x_i)+h(x_i)$.

For linear function $h_k(x)$, $k = 1, 2, 3.$ we check is it satisfied max principle:

$$h_k(x_{i+\frac{1}{2}}) \in (h(x_i), h(x_{i+1})) \qquad and \qquad h_k(x_{i-\frac{1}{2}}) \in (h(x_{i-1}), h(x_i)).$$

First we check is function h_1 satisfied max principle if not then we check the other two functions. The function that satisfy max principle can be approximation on the interval. If non of the $h_k(x)$, $k = 1, 2, 3.$ is good approximation then we can for that interval remain first order approximation. The method that has been explained is called **min mod**. On the border intervals we remain first order of approximation because no matter how we approximate function on this intervals total variational will increase. Picture below shows how that method works. Using this method we make total variation smaller.



Second order approximation of the flux we obtain with:

$$\phi^n_{i+\frac{1}{2}} = \phi(\bar{w}^n_i, \bar{w}^n_{i+1}) = \frac{1}{2}(f_x((\bar{w}^n_i)_R) + f_x((\bar{w}^n_{i+1})_L) + \mid \xi \mid ((\bar{w}^n_i)_R - (\bar{w}^n_{i+1})_L))$$

where $(\bar{w}^n_i)_R$ marks value of the corresponding approximated function in the right point of the interval i. And $(\bar{w}^n_{i+1})_L$ marks value of the corresponding approximated function in the left point of the interval $i+1$.

Second order approximation scheme - for this scheme we are using predictor and corrector.

$$\bar{w}_i^* = \bar{w}_i^n + \frac{\frac{1}{2}(t^{n+1} - t^n)}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}}[\phi(\bar{w}_i^n, \bar{w}_{i+1}^n) - \phi(\bar{w}_{i-1}^n, \bar{w}_i^n)]$$

$$\bar{w}_i^{n+1} = \bar{w}_i^n + \frac{t^{n+1} - t^n}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}}[\phi(\bar{w}_i^*, \bar{w}_{i+1}^*) - \phi(\bar{w}_{i-1}^*, \bar{w}_i^*)]$$

Mathematical properties of 2-D equation.

If we observe functions $f_x(h, hu)$ and $f_y(h, hu)$ it is easy to calculate eigenvalues of the Jacobean of f_x and f_y.

$$A = \frac{\partial f_x}{\partial w} = \begin{pmatrix} 0 & 1 & 0 \\ -u^2 + gh & 2u & 0 \\ -uv & v & u \end{pmatrix}$$

$$B = \frac{\partial f_y}{\partial w} = \begin{pmatrix} 0 & 1 & 0 \\ -uv & u & v \\ -v^2 + gh & 2v & 0 \end{pmatrix}$$

The matrix A is easily shown to have real eigenvalues and linearly independent eigenvectors. In particular, the eigenvalues of A are given:

$$\lambda_1 = u + \sqrt{gh} \qquad \lambda_2 = u - \sqrt{gh} \qquad \lambda_3 = u.$$

and for B we have the same:

$$\lambda_1 = v + \sqrt{gh} \qquad \lambda_2 = v - \sqrt{gh} \qquad \lambda_3 = v.$$

Numerical approximation of 2-D equation(Finite volumes).

Like in 1-D case, we discretaze domain with uniform mesh on the space domain and uniform mesh on the time domain. On the space domain we have intervals: $\left(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right) \times \left(y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}\right)$. We integrate equation on the interval $\left(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right) \times \left(y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}\right) \times \left(t^n, t^{n+1}\right)$:

$$\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{t^n}^{t^{n+1}} \frac{\partial}{\partial t} w dt dx dy + \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{t^n}^{t^{n+1}} \frac{\partial}{\partial x} f_x(w) dt dx dy +$$

$$+ \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{t^n}^{t^{n+1}} \frac{\partial}{\partial y} f_y(w) dt dx dy = 0$$

we get:

$$\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (w(t^{n+1}, x, y) - w(t^n, x, y)) dx dy + \int_{t^n}^{t^{n+1}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} (f_x(w(t, x_{i+\frac{1}{2}}, y)) - f_x(w(t, x_{i-\frac{1}{2}}, y))) dy dt +$$

$$+ \int_{t^n}^{t^{n+1}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (f_y(w(t, x, y_{j+\frac{1}{2}})) - f_y(w(t, x, y_{j-\frac{1}{2}}))) dx dt = 0$$

Shortly written:

$$(t^{n+1} - t^n) h_y (\phi^n_{i+\frac{1}{2},j} - \phi^n_{i-\frac{1}{2},j}) + (t^{n+1} - t^n) h_x (\phi^n_{i,j+\frac{1}{2}} - \phi^n_{i,j-\frac{1}{2}}) + h_x h_y (\bar{w}_i^{n+1} - \bar{w}_i^n) = 0$$

where:

$$\bar{w}_i^n = \frac{1}{h_x h_y} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (w(t^n, x, y) dx dy$$

$$\phi^n_{i+\frac{1}{2},j} = \frac{1}{(t^{n+1}-t^n)h_y}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\int_{t^n}^{t^{n+1}}(f_x(w(t,x_{i+\frac{1}{2}},y))dtdy$$

$$\phi^n_{i,j+\frac{1}{2}} = \frac{1}{(t^{n+1}-t^n)h_x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{t^n}^{t^{n+1}}(f_x(w(t,x,y_{j+\frac{1}{2}}))dtdx$$

where $h_x = x_{i-\frac{1}{2}} - x_{i+\frac{1}{2}}$, $h_y = y_{j-\frac{1}{2}} - y_{j+\frac{1}{2}}$.

If we can approximate flux $\phi^n_{i,j+\frac{1}{2}}$ such that $\phi^n_{i,j+\frac{1}{2}} = \phi(\bar{w}^n_{i,j}, \bar{w}^n_{i,j+1})$, and $\phi^n_{i+\frac{1}{2},j} = \phi(\bar{w}^n_{i,j}, \bar{w}^n_{i+1,j})$ the we have obtained scheme.

Approximation of the flux:

Scheme is stable if we approximate:

$$\phi^n_{i+\frac{1}{2},j} = \phi(\bar{w}^n_{i,j}, \bar{w}^n_{i+1,j}) = \frac{1}{2}(f_x(\bar{w}^n_{i,j}) + f_x(\bar{w}^n_{i+1,j}) + \mid \xi_1 \mid (\bar{w}^n_{i,j} - \bar{w}^n_{i+1,j}))$$

and

$$\phi^n_{i,j+\frac{1}{2}} = \phi(\bar{w}^n_{i,j}, \bar{w}^n_{i,j+1}) = \frac{1}{2}(f_y(\bar{w}^n_{i,j}) + f_y(\bar{w}^n_{i,j+1}) + \mid \xi_2 \mid (\bar{w}^n_{i,j} - \bar{w}^n_{i,j+1}))$$

where $\mid \xi_1 \mid \geq max(\mid \lambda \mid_{f_x})_{w\in\{\bar{w}^n_{i,j},\bar{w}^n_{i+1,j}\}}$ and $\mid \xi_2 \mid \geq max(\mid \lambda \mid_{f_y})_{w\in\{\bar{w}^n_{i,j},w^n_{i,j+1}\}}$.

references:

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