Reduced Basis Approach for Nonlinear Elasticity

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1 Derivation of the equations

1.1 Preliminaries

1.1.1 Basic notions

Let there be a bounded, open, connected subset Ω of $\mathbb{R}^3$ with sufficiently smooth boundary. $\bar{\Omega}$ represents the volume occupied by a body before it is deformed and is called **reference configuration**. A **deformation** of the reference configuration $\bar{\Omega}$ is a vector field

$$\boldsymbol{\varphi} : \bar{\Omega} \to \mathbb{R}^3$$

smooth enough, injective except possibly the boundary $\partial\Omega$ of the set Ω and orientation preserving.

$$\nabla\boldsymbol{\varphi} = \begin{pmatrix} \partial_1\varphi_1 & \partial_2\varphi_1 & \partial_3\varphi_1 \\ \partial_1\varphi_2 & \partial_2\varphi_2 & \partial_3\varphi_2 \\ \partial_1\varphi_3 & \partial_2\varphi_3 & \partial_3\varphi_3 \end{pmatrix}, \qquad \textit{with } \partial_i := \frac{\partial}{\partial x_i}$$

is called the **deformation gradient**. The orientation preserving condition leads to the condition

$$\det \nabla\boldsymbol{\varphi}(\mathbf{x}) > 0 \ \forall \mathbf{x} \in \bar{\Omega}$$

. This implies that any deformation gradient is invertible i.e. $\nabla\boldsymbol{\varphi}(\mathbf{x})^{-1}$ exists for all $\mathbf{x} \in \bar{\Omega}$.

One may write

$$\boldsymbol{\varphi} = \mathbf{id} + \mathbf{u}$$

with

$$\mathbf{u} : \bar{\Omega} \to \mathbb{R}^3$$

the **displacement**. We have $\nabla\boldsymbol{\varphi} = \mathbf{I} + \nabla\mathbf{u}$.

We will call $\varphi(\bar{\Omega})$ the deformed configuration and denote $\mathbf{x}^{\varphi} := \boldsymbol{\varphi}(\mathbf{x}) \ \forall \mathbf{x} \in \bar{\Omega}$.

For any volume element $dx^{\varphi} := \boldsymbol{\varphi}(dx)$ we have

$$dx^{\varphi} = \det \nabla\boldsymbol{\varphi}(x) dx \ \ (\det \nabla\boldsymbol{\varphi} > 0)$$

and therefore for any non-measure zero subset $A \subset \bar{\Omega}$

$$vol\ A = \int_A dx, \qquad vol\ A^{\varphi} = \int_{A^{\varphi}} dx^{\varphi} = \int_A \det \nabla\boldsymbol{\varphi} dx$$

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1.1.2 Notes about 2^{nd}-order tensors

We will denote the set of all 2^{nd}-order tensors by $\mathbb{M}^3$. Let $\mathbf{T} \in \mathbb{M}^3$. Then we define the divergence of $\mathbf{T}$ by

$$\mathbf{div}\ \mathbf{T} := \partial_j T_{ij}\mathbf{e}_i = \begin{pmatrix} \partial_1 T_{11} + \partial_2 T_{12} + \partial_3 T_{13} \\ \partial_1 T_{21} + \partial_2 T_{22} + \partial_3 T_{23} \\ \partial_1 T_{31} + \partial_2 T_{32} + \partial_3 T_{33} \end{pmatrix}$$

and we have by Gauß's theorem

$$\int_\Omega \mathbf{div}\ \mathbf{T} dx = \int_{\partial\Omega} \mathbf{T}\cdot\mathbf{n} dx$$

where $\mathbf{n}$ denotes the unit outer normal vector of $\partial\Omega$. In the same way we define $\mathbf{div}^\varphi\ \mathbf{T}^\varphi = \partial_j^\varphi \mathbf{T}_{ij}^\varphi \mathbf{e}_i, \quad \partial_j^\varphi := \frac{\partial}{\partial x_j^\varphi}$ and have $\int_{\Omega^\varphi} \mathbf{div}^\varphi\ \mathbf{T}^\varphi dx^\varphi = \int_{\partial\Omega^\varphi} \mathbf{T}^\varphi \cdot \mathbf{n}^\varphi dx^\varphi$. For any tensor $\mathbf{T}^\varphi(\mathbf{x}^\varphi) \in \mathbb{M}^3$ defined at a point $\mathbf{x}^\varphi = \boldsymbol{\varphi}(\mathbf{x})$ we define the **Piola transform** of that tensor by

$$\mathbf{T}(\mathbf{x}) := \det\nabla\boldsymbol{\varphi}(\mathbf{x})\mathbf{T}^\varphi(\mathbf{x}^\varphi)\nabla\boldsymbol{\varphi}^{-T} = \mathbf{T}^\varphi(\mathbf{x}^\varphi)\mathbf{cof}\boldsymbol{\varphi}(\mathbf{x})$$

equivalently we have $\mathbf{T}^\varphi(\mathbf{x}^\varphi) = (\det\nabla\boldsymbol{\varphi}(\mathbf{x}))^{-1}\,\mathbf{T}(\mathbf{x})\nabla\boldsymbol{\varphi}^T$.

1.1.3 theorem (properties of the Piola transform)

Let $\mathbf{T}:\Omega\to\mathbb{M}^3$ *denote the Piola transform of* $\mathbf{T}^\varphi : \bar\Omega^\varphi \to \mathbb{M}^3$. *Then*

$$\begin{array}{ccc} \mathbf{div}\ \mathbf{T}(\mathbf{x}) & = & (\det\nabla\boldsymbol{\varphi}(x))\mathbf{div}^\varphi\mathbf{T}^\varphi(x^\varphi),\ \forall x^\varphi = \boldsymbol{\varphi}(x),\ x\in\bar\Omega \\ \mathbf{T}(x)\mathbf{n}\ da & = & \mathbf{T}^\varphi(x^\varphi)\mathbf{n}^\varphi da^\varphi,\ \forall x^\varphi = \boldsymbol{\varphi}(x),\ x\in\partial\Omega \\ da^\varphi & = & |\mathbf{Cof}\ \nabla\boldsymbol{\varphi}\mathbf{n}|\, da = \det\nabla\boldsymbol{\varphi}(x)\left|\nabla\boldsymbol{\varphi}(x)^{-T}\mathbf{n}\right| da \end{array}$$

The proof uses the **Piola identity** $\mathbf{div}\{(\det\nabla\boldsymbol{\varphi})\nabla\boldsymbol{\varphi}^{-T}\} = \mathbf{divCof}\nabla\boldsymbol{\varphi} = \mathbf{0}$.

1.1.4 Rigid deformations

The two tensors $\mathbf{C} := \nabla\boldsymbol{\varphi}^T\nabla\boldsymbol{\varphi}$ and $\mathbf{B} := \nabla\boldsymbol{\varphi}\nabla\boldsymbol{\varphi}^T$ are called **right Cauchy-Green strain tensor** and **left Cauchy-Green strain tensor**. A deformation $\boldsymbol{\varphi}$ that has the form

$$\boldsymbol{\varphi}(\mathbf{x}) = \mathbf{a} + \mathbf{Q}\mathbf{x},\ \mathbf{a}\in\mathbb{R}^3,\ \mathbf{Q}\in\mathbb{O}^3_+,\ \forall\mathbf{x}\in\bar\Omega$$

,where $\mathbb{O}^3_+$ denotes the set of orthogonal matrices of order 3 with determinant equal to +1, is called a **rigid deformation**. Rigid deformations induce no strain and it can be shown, that for all $\boldsymbol{\varphi}\in\mathcal{C}^1(\Omega;\mathbb{R}^n)$ that satisfy $\nabla\boldsymbol{\varphi}(x)^T\nabla\boldsymbol{\varphi}(x) = \mathbf{I},\ \forall x\in\Omega$ exists a vector $a\in\mathbb{R}^n$ and an orthogonal matrix $\mathbf{Q}\in\mathbb{O}^n$ such that

$$\boldsymbol{\varphi}(x) = \mathbf{a} + \mathbf{Q}\mathbf{x},\ \forall x\in\Omega$$

1.2 Equations of equilibrium

1.2.1 Applied forces and Cauchy stress tensor

There are two types of **applied forces**:
applied body forces, defined by a vector field

$$\mathbf{f}^\varphi := \Omega^\varphi \to \mathbb{R}^3$$

that is called the **density of the applied body force** per unit volume in the deformed configuration and

applied surface forces, defined by a vector field

$$\mathbf{g}^{\varphi} : \Gamma_1^{\varphi} \to \mathbb{R}^3$$

on a da^{φ}- measurable subset Γ_1^{φ} of the boundary $\Gamma^{\varphi} := \partial\Omega^{\varphi}$, that is called the **density of the applied surface force** per unit area in the deformed configuration. Clearly applied body forces act on the volume of a body or equally on the mass of a body whereas applied surface forces act on the surface of a body and cannot penetrate these. Typical body forces are gravity and electrical field and typical surface forces are stress and pressure. Since the applied surface forces can just be defined on a da^{φ}- measurable subset of the boundary Γ^{φ} we define $\Gamma_0^{\varphi} := \Gamma^{\varphi} - \Gamma_1^{\varphi}$.

1.2.2 Axiom (stress principle of Euler and Cauchy)

Consider a body occupying a deformed configuration $\bar{\Omega}^{\varphi}$, and subjected to applied forces represented by densities $\mathbf{f}^{\varphi} := \Omega^{\varphi} \to \mathbb{R}^3$ and $\mathbf{g}^{\varphi} : \Gamma_1^{\varphi} \to \mathbb{R}^3$. Then there exists a vector field

$$\boxed{\mathbf{t}^{\varphi} : \bar{\Omega}^{\varphi} \times S_1 \to \mathbb{R}^3, \ \textit{where}\ S_1 = \left\{v \in \mathbb{R}^3;\ |v| = 1\right\}}$$

such that:

1. *For any subdomain A^{φ} of $\bar{\Omega}^{\varphi}$, and at any point $x^{\varphi} \in \Gamma_1 \cap \partial A^{\varphi}$ where the unit outer normal vector $\mathbf{n}^{\varphi}$ to $\Gamma_1 \cap \partial A^{\varphi}$ exists*

$$\boxed{\mathbf{t}^{\varphi}(x^{\varphi}, \mathbf{n}^{\varphi}) = \mathbf{g}^{\varphi}(x^{\varphi})}$$

2. ***Axiom of force balance**: For any subdomain A^{φ} of $\bar{\Omega}^{\varphi}$,*

$$\boxed{\int_{A^{\varphi}} \mathbf{f}^{\varphi}(x^{\varphi}) dx^{\varphi} + \int_{\partial A^{\varphi}} \mathbf{t}^{\varphi}(x^{\varphi}, \mathbf{n}^{\varphi}) da^{\varphi} = \mathbf{0}}$$

   where $\mathbf{n}^{\varphi}$ *denotes the unit outer normal vector along ∂A^{φ}.*

3. ***Axiom of moment balance**: For any subdomain A^{φ} of Ω^{φ},*

$$\boxed{\int_{A^{\varphi}} \mathbf{x}^{\varphi} \times \mathbf{f}^{\varphi}(x^{\varphi}) dx^{\varphi} + \int_{\partial A^{\varphi}} \mathbf{x}^{\varphi} \times \mathbf{t}^{\varphi}(x^{\varphi}, \mathbf{n}^{\varphi}) da^{\varphi} = \mathbf{0}}$$

   The Euler's and Cauchy's stress principle asserts the existence of elementary **surface forces** $\mathbf{t}^{\varphi}(x^{\varphi}, \mathbf{n}^{\varphi}) da^{\varphi}$ along the boundaries ∂A^{φ} of all subsets A^{φ}, that those surface forces just depend on the normal vector $\mathbf{n}^{\varphi}$ and that any subdomain $A^{\varphi} \subset \bar{\Omega}^{\varphi}$ is in *static equilibrium*, in the sense, that the sum of all forces is equally zero. The vector $\mathbf{t}^{\varphi}(x^{\varphi}, \mathbf{n}^{\varphi})$ is called **Cauchy stress vector**.

1.2.3 Theorem (Cauchy's theorem)

Assume that the applied body force density $\mathbf{f}^\varphi : \bar{\Omega}^\varphi \to \mathbb{R}^3$ *is continuous, and that the Cauchy stress vector field*

$$\mathbf{t}^\varphi : (x^\varphi, \mathbf{n}^\varphi) \in \bar{\Omega}^\varphi \times S_1 \to \mathbf{t}(x^\varphi, \mathbf{n}^\varphi) \in \mathbb{R}^3$$

is continuously differentiable with respect to the variable $x^\varphi \in \bar{\Omega}^\varphi$ *for each* $\mathbf{n} \in S_1$ *and continuous with respect to the variable* $\mathbf{n}^\varphi \in S_1$ *for each* $x^\varphi \in \bar{\Omega}^\varphi$*. Then the axioms of force and moment balance imply that there exists a continuously differentiable tensor field*

$$\mathbf{T}^\varphi : x^\varphi \in \bar{\Omega}^\varphi \to \mathbf{T}^\varphi(x^\varphi) \in \mathbb{M}^3,$$

such that the Cauchy stress vector satisfies

$$\boxed{\mathbf{t}^\varphi(x^\varphi, \mathbf{n}) = \mathbf{T}^\varphi \cdot \mathbf{n} \text{ for all } x^\varphi \in \bar{\Omega}^\varphi \text{ and all } \mathbf{n} \in S_1}$$

and such that

$$\boxed{-\mathbf{div}^\varphi \mathbf{T}^\varphi(x^\varphi) = \mathbf{f}^\varphi(x^\varphi) \quad \forall x^\varphi \in \Omega^\varphi}$$

$$\boxed{\mathbf{T}^\varphi(x^\varphi) = \mathbf{T}^\varphi(x^\varphi)^T \quad \forall x^\varphi \in \bar{\Omega}^\varphi}$$

$$\boxed{-\mathbf{T}^\varphi(x^\varphi)\mathbf{n}^\varphi = \mathbf{g}^\varphi(x^\varphi) \quad \forall x^\varphi \in \Gamma_1^\varphi}$$

where $\mathbf{n}^\varphi$ *is the unit outer normal vector along* Γ_1^φ*.*

The proof uses the stress principle of Euler and Cauchy and a particular subset of Ω^φ.

The symmetric tensor $\mathbf{T}^\varphi(x^\varphi)$ is called the **Cauchy stress tensor** at the point $x^\varphi \in \bar{\Omega}^\varphi$.

1.3 Principle of virtual work in the deformed configuration

1.3.1 Theorem (principle of virtual work)

The boundary value problem

$$\begin{aligned} -\mathbf{div}^\varphi \mathbf{T}^\varphi &= \mathbf{f}^\varphi \textit{ in } \Omega \\ \mathbf{T}^\varphi &= \mathbf{g}^\varphi \textit{ on } \Gamma_1^\varphi \end{aligned}$$

is formally equivalent to the variational equations

$$\boxed{\int_{\Omega^\varphi} \mathbf{T}^\varphi : \nabla^\varphi \boldsymbol{\theta}^\varphi dx^\varphi = \int_{\Omega^\varphi} \mathbf{f}^\varphi \cdot \boldsymbol{\theta}^\varphi dx^\varphi + \int_{\Gamma_1^\varphi} \mathbf{g}^\varphi \cdot \boldsymbol{\theta}^\varphi dx^\varphi}$$

valid for all smooth enough vector fields: $\boldsymbol{\theta}^\varphi : \Omega^\varphi \to \mathbb{R}^3$ *that satisfy*

$$\boldsymbol{\theta}^\varphi = \mathbf{0} \textit{ on } \Gamma_0^\varphi := \Gamma^\varphi - \Gamma_1^\varphi.$$

The proof uses mainly the Green's formula.

The integral equation is called the **principle of virtual work in the deformed configuration**. The principle of virtual work can be directly deduced from the axiom of force balance. Since the requirement of regularity of $\mathbf{T}^\varphi$ in the integral equation is less than in the equations of equilibrium, the requirements of smoothness of $\mathbf{T}^\varphi$ is naturally just very mild, since it is only required that all integrals make sense.

1.4 Equations in the reference configuration

1.4.1 Definition (Piola-Kirchhoff stress tensor)

The Piola transform of the Cauchy stress tensor $\mathbf{T}^\varphi$ *is called* ***first Piola-Kirchhoff stress tensor****.*

With the first Piola-Kirchhoff stress tensor we get

$$\mathbf{divT} = (\det \nabla\varphi(x))\mathbf{div}^\varphi \mathbf{T}^\varphi(x^\varphi), \qquad x^\varphi = \varphi(x).$$

While the Cauchy stress tensor $\mathbf{T}^\varphi$ is symmetric the first Piola-Kirchhoff stress tensor is not symmetric in general anymore. Instead one has:

$$\mathbf{T}(x)^T = \nabla\varphi^{-1}\mathbf{T}(x)\nabla\varphi(x)^{-T}.$$

If one wants to define a symmetric tensor in the reference configuration, one might define the **second Piola-Kirchhoff stress tensor** $\mathbf{\Sigma}(x)$:

$$\begin{aligned}\mathbf{\Sigma}(x) &= \nabla\varphi(x)^{-1}\mathbf{T}(x) = (\det \nabla\varphi(x))\nabla\varphi(x)^{-1}\mathbf{T}^\varphi(x^\varphi)\nabla\varphi(x)^{-T} \\ & \quad x^\varphi = \varphi(x).\end{aligned}$$

Further I will not look at the second Piola-Kirchhoff stress tensor.

Transforming the equations of equilibrium to the reference configuration lead to the **equations of equilibrium in the reference configuration**

$$\begin{aligned}-\mathbf{div}\ \mathbf{T}(x) &= \mathbf{f}(x),\ x \in \Omega \\ \nabla\varphi(x)\mathbf{T}(x)^T &= \mathbf{T}(x)\nabla\varphi(x)^T,\ x \in \Omega \\ \mathbf{T}(x)\mathbf{n} &= \mathbf{g}(x),\ x \in \Gamma_1\end{aligned}$$

with $\mathbf{f}\ dx = \mathbf{f}^\varphi\ dx^\varphi,\ \mathbf{g}\ da = \mathbf{g}^\varphi\ da^\varphi$ and the **principle of virtual work in the reference configuration**

$$\int_\Omega \mathbf{T} : \nabla\boldsymbol{\theta}\ dx = \int_\Omega \mathbf{f}\cdot\boldsymbol{\theta}\ dx + \int_{\Gamma_1} \mathbf{g}\cdot\boldsymbol{\theta}\ da$$

Definition

An applied body force with density $\mathbf{f} : \Omega \to \mathbb{R}^3$ *in the reference configuration is called* ***conservative****, if the integral*

$$\int_\Omega \mathbf{f}(x)\cdot\boldsymbol{\theta}(x)\ dx = \int_\Omega \hat{\mathbf{f}}(x,\varphi(x))\cdot\boldsymbol{\theta}(x)\ dx$$

that appears in the principle of virtual work in the reference configuration can also be written as the Gâteaux derivative

$$\boxed{F'(\varphi)\boldsymbol{\theta} = \int_\Omega \hat{\mathbf{f}}(x, \varphi(x)) \cdot \boldsymbol{\theta}(x)\ dx}$$

of a functional of the form

$$F : \{\boldsymbol{\psi} : \bar{\Omega} \to \mathbb{R}^3\} \to F(\boldsymbol{\psi}) = \int_\Omega \hat{F}(x, \boldsymbol{\psi}(x))\ dx$$

. $\hat{F} : \Omega \times \mathbb{R}^3 \to \mathbb{R}$ *is then called the* ***potential*** *of the applied body force. In the same way one calls a surface force with density* $\mathbf{g} : \Gamma_1 \to \mathbb{R}^3$ *conservative if the integral*

$$\int_{\Gamma_1} \mathbf{g}(x) \cdot \boldsymbol{\theta}(x)\ da = \int_{\Gamma_1} \hat{\mathbf{g}}(x, \nabla\varphi(x)) \cdot \boldsymbol{\theta}(x)\ da$$

can also be written as the Gâteaux Derivative

$$\boxed{G'(\varphi)\boldsymbol{\theta} = \int_{\Gamma_1} \hat{\mathbf{g}}(x, \nabla\varphi(x)) \cdot \boldsymbol{\theta}(x)\ da}$$

of a Functional of the form

$$G : \{\boldsymbol{\psi} : \bar{\Omega} \to \mathbb{R}^3\} \to G(\boldsymbol{\psi}) = \int_\Omega \hat{G}(x, \boldsymbol{\psi}(x), \nabla\boldsymbol{\psi}(x)))\ da$$

. $\hat{G} : \Gamma_1 \times \mathbb{R}^3 \times \mathbb{M}^3_+ \to \mathbb{R}$ *is then called the* ***potential*** *of the applied surface force.*

1.5 Elastic material

1.5.1 Definition (elastic material)

A material is called ***elastic*** *if there exists a mapping*

$$\hat{\mathbf{T}}^D : (x, \mathbf{F}) \in \bar{\Omega} \times \mathbb{M}^3_+ \to \hat{\mathbf{T}}^D(x, \mathbf{F}) \in \mathbb{S}^3,$$

called the ***response function for the Cauchy stress****, such that for any deformation and any point in the deformed configuration the* ***constitutive equation***

$$\mathbf{T}^\varphi(x^\varphi) = \hat{\mathbf{T}}^D(x, \nabla\varphi(x)),\ x^\varphi = \varphi(x).$$

Here $\mathbb{S}^3$ denotes the set of all symmetric matrices of order 3 and D in the exponent of $\hat{\mathbf{T}}$ reminds us that the response function is defined in the *deformed* configuration. By definition one has also the Piola transform of $\hat{\mathbf{T}}^D$ given by

$$\hat{\mathbf{T}} = (\det \mathbf{F})\hat{\mathbf{T}}^D(x, \mathbf{F})\mathbf{F}^{-T}$$

and we have

$$\mathbf{T}(x) = \hat{\mathbf{T}}(x, \nabla\varphi(x)),\ \forall x \in \bar{\Omega}$$

called the **response function for the first Piola-Kirchhoff stress**.

1.5.2 Definition (homogeneous material)

A material in the reference configuration is called homogeneous, if its respnse function is independent of the particular point $x \in \bar{\Omega}$ and the constitutive equation takes up the form

$$\mathbf{T}^{\varphi}(x^{\varphi}) = \hat{\mathbf{T}}^{D}(\nabla\varphi(x)), \ \forall x^{\varphi} = \varphi(x) \in \bar{\Omega}^{\varphi}$$

1.6 Hyperelastic material

1.6.1 Definition (hyperelastic material)

*An elastic material with response function $\hat{\mathbf{T}} : \bar{\Omega} \times \mathbb{M}^3_+ \to \mathbb{M}^3$ is called **hyperelastic** if there exists a function*

$$\hat{\mathbf{W}} : \bar{\Omega} \times \mathbb{M}^3_+ \to \mathbb{R}$$

differentiable with respect to the variable $\mathbf{F} \in \mathbb{M}^3_+$ for each $x \in \bar{\Omega}$, such that

$$\hat{\mathbf{T}}(x, \mathbf{F}) = \frac{\partial \hat{\mathbf{W}}}{\partial \mathbf{F}}(x, \mathbf{F}), \ \forall x \in \Omega, \mathbf{F} \in \mathbb{M}^3_+$$

*. Then the function $\hat{\mathbf{W}}$ is called the **stored energy function**.*

1.6.2 Theorem (minimal property)

Let there be given a hyperelastic material subjected to conservative applied body forves and conservative applied surface forces. Then the equations

$$\begin{aligned} -\mathbf{div}\frac{\partial \hat{\mathbf{W}}}{\partial \mathbf{F}}(x, \nabla\varphi(x)) &= \hat{\mathbf{f}}(x, \varphi(x)), \ x \in \Omega \\ \frac{\partial \hat{\mathbf{W}}}{\partial \mathbf{F}}(x, \nabla\varphi(x))\mathbf{n} &= \hat{\mathbf{g}}(x, \nabla\varphi(x)), \ x \in \Gamma_1 \end{aligned}$$

are formally equivalent to the equations

$$\boxed{I'(\varphi)\boldsymbol{\theta} = 0,}$$

for all smooth maps $\boldsymbol{\theta} : \bar{\Omega} \to \mathbb{R}^3$ that vanish on Γ_0, where the functional I is defined for smooth enough mappiongs $\boldsymbol{\psi} : \bar{\Omega} \to \mathbb{R}^3$ by

$$\boxed{I(\boldsymbol{\psi}) = \int_{\Omega} \hat{\mathbf{W}}(x, \nabla\boldsymbol{\psi}(x)) \ dx - \{F(\boldsymbol{\psi}) + G(\boldsymbol{psi})\}}$$

In our case $\hat{\mathbf{W}}$ was defined

$$\hat{\mathbf{W}}(\mathbf{F}) = \alpha\|\mathbf{F}\|^2 + \beta\|\mathbf{Cof}\mathbf{F}\|^2 + \delta\|\det\mathbf{F}\|^2 - \delta\ln(\det\mathbf{F}).$$

for given α, β, γ *and* δ.

2 Numerics

2.1 Finite element method with Newton algorithm

We want to solve the problem

$$\int_\Omega \frac{\partial W}{\partial F}(\nabla \phi_S) : \nabla \theta_i \; dx = \int_\Omega f \cdot \theta_i \; dx, \qquad \forall \theta_i \in \Theta_S$$

for $\Omega = [0,5] \times [0,1]$ where Θ_S denotes the finite element subspace. We have chosen $\theta_i(x,y)$ piecewise linear in both koordinates $\left(P^1\right)$ and a basis that has the structure:

$$\theta_i(x_j) = \begin{cases} \delta_{ij} & for\ i = 1 \ldots m \\ \delta_{(i-m)j} & for\ i = m+1 \ldots n \end{cases}$$

Since these equations are nonlinear, we used a Newton algorithm for the ϕ_i in $\phi_S(x,y) = \sum_{i=1}^n \phi_i \theta_i(x,y)$. Further we will write $\phi_S(x,y) = id(x,y) + u_S(x,y)$. The Newton algorithm has the form:

$$\begin{cases} DG(\underline{\phi}^k)\triangle\phi & = G(\underline{\phi}^k) \qquad \underline{\phi}^k = (\phi_1^k \ldots \phi_n^k)^t \\ \underline{\phi}^{k+1} & = \underline{\phi}^k - \triangle\phi \end{cases}$$

with

$$\begin{aligned} \left[DG(\underline{\phi}^k)\right]_{ij} &= \int_\Omega \frac{\partial}{\partial \phi_j} \left(\frac{\partial W}{\partial F}(\nabla \phi) \right) : \nabla \theta_i \; dx \\ \left[G(\underline{\phi}^k)\right]_i &= \int_\Omega \frac{\partial W}{\partial F}(\nabla \phi_S) : \nabla \theta_i \; dx - \int_\Omega f \cdot \theta \; dx \end{aligned}$$

Starting point was $u_i^0 = 0, \quad \forall i = 1 \ldots n$. In detail $\left[G(\underline{\phi}^k)\right]_i$ was:

$$\int_\Omega B : \nabla \theta_i \; dx - \int_\Omega f \cdot \theta_i \; dx$$

with

$$\begin{aligned} B &= \begin{bmatrix} 2a_{11}(\alpha+\beta) + 2\gamma a_{22} \det A - \dfrac{\delta a_{22}}{\det A} & 2a_{12}(\alpha+\beta) - 2\gamma a_{21} \det A + \dfrac{\delta a_{21}}{\det A} \\ 2a_{21}(\alpha+\beta) - 2\gamma a_{12} \det A + \dfrac{\delta a_{12}}{\det A} & 2a_{22}(\alpha+\beta) + 2\gamma a_{11} \det A - \dfrac{\delta a_{11}}{\det A} \end{bmatrix} \\ A &= \nabla \phi_S(x,y) \end{aligned}$$

and $\left[DG(\underline{\phi}^k)\right]_{ij}$:

$$\int_\Omega (c_{11} \cdot W_{11} + c_{12} \cdot W_{12} + c_{21} \cdot W_{21} + c_{22} \cdot W_{22}) : \nabla \theta_i \; dx$$

with

$$W_{ij} = \frac{\partial W}{\partial F_{ij}} \quad i.e.$$

$$W_{11} = \begin{bmatrix} 2(\alpha+\beta+\gamma a_{22}^2) + \dfrac{\delta a_{22}^2}{(\det A)^2} & -2\gamma a_{21}a_{22} - \dfrac{\delta a_{21}a_{22}}{(\det A)^2} \\ -2\gamma a_{22}a_{12} - \dfrac{\delta a_{12}a_{22}}{(\det A)^2} & 2\gamma(a_{11}a_{22} + \det A) + \delta\dfrac{a_{11}a_{22} - \det A}{(\det A)^2} \end{bmatrix}$$

$$W_{12} = \begin{bmatrix} -2\gamma a_{22}a_{21} - \dfrac{\delta a_{21}a_{22}}{(\det A)^2} & 2(\alpha+\beta+\gamma a_{21}^2) + \dfrac{\delta a_{21}^2}{(\det A)^2} \\ 2\gamma(a_{21}a_{12} - \det A) + \delta\dfrac{a_{12}a_{21} + \det A}{(\det A)^2} & -2\gamma a_{21}a_{11} - \dfrac{\delta a_{11}a_{21}}{(\det A)^2} \end{bmatrix}$$

$$W_{21} = \begin{bmatrix} -2\gamma a_{22}a_{21} - \dfrac{\delta a_{12}a_{22}}{(\det A)^2} & 2\gamma(a_{21}a_{12} - \det A) + \delta\dfrac{a_{12}a_{21} + \det A}{(\det A)^2} \\ 2(\alpha+\beta+\gamma a_{12}^2) + \dfrac{\delta a_{12}^2}{(\det A)^2} & -2\gamma a_{21}a_{11} - \dfrac{\delta a_{11}a_{21}}{(\det A)^2} \end{bmatrix}$$

$$W_{22} = \begin{bmatrix} 2\gamma(a_{11}a_{22} + \det A) + \delta\dfrac{a_{11}a_{22} - \det A}{(\det A)^2} & -2\gamma a_{21}a_{11} - \dfrac{\delta a_{11}a_{21}}{(\det A)^2} \\ -2\gamma a_{12}a_{11} - \dfrac{\delta a_{11}a_{12}}{(\det A)^2} & 2(\alpha+\beta+\gamma a_{11}^2) + \dfrac{\delta a_{11^2}}{(\det A)^2} \end{bmatrix}$$

$$C = \nabla\theta_j(x,y)$$

To deal with the boundary conditions we added a penelization term $\frac{1}{\epsilon}\sum_{j=1}^{n} u_j \int_\omega \theta_i \cdot \theta_j$, where ω is the boundary Γ_1.

2.2 Reduced basis approach

The basic idea is to precompute solutions and use those solutions as basis functions in our finite elements solution subspace i.e. $\Theta_S = U_r = span\{u_i, i = 1 \dots I\}$, where u_i are the displacements of the solutions ϕ_S.
We end up solving

$$\int_\Omega \frac{\partial W}{\partial F}(Id + \nabla u_r) : \nabla\delta u_r \; dx = \int_\Omega f \cdot \delta u_r \; dx, \; \forall \delta u_r \in U_r$$

Within the Newton iteration one can precompute many values that are needed and load them during running time from a file. This is not done in the code, why running times are not representative. In the code we haven't used the supposed way to calculate basis functions. Instead we were using different linear functions f for the RHS, i.e.

$$f(x,y) = \begin{pmatrix} a_1 x + a_2 y + a_3 \\ b_1 x + b_2 y + b_3 \end{pmatrix}, \quad M = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

For the calculating the solutions we have set all the koordinates of M equal to zero except one. So for the first six solution we were setting

$$\begin{aligned} M_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, M_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \\ M_4 &= \begin{pmatrix} 0 & 0 & 0 \\ .8 & 0 & 0 \end{pmatrix}, M_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, M_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

The value for b_1 is different because giving the values $\alpha = \beta = \gamma = 1, \ \delta = 5$, the Newton iteration doesn't converge for $b_1 = 1$. For the other solutions we have decreased the koefficients in M. $\{a_1, a_2, a_3, b_2, b_3\}$ toke up the values $\{1, 0.8, 0.6, 0.4, 0.2\}$ and b_1 the values $\{0.8, 0.7, 0.6, 0.4, 0.2\}$.

2.3 explaining the code:

2.3.1 FE.m

Variables

- Nodes - Saves the x and y coordinate of the nodes. So $Nodes(:,i) = \begin{pmatrix} x_i & y_i & flag \end{pmatrix}$
- ConnTable - Saves the connectivity table
- EdgeTable - not used
- Phi - saves the values for the $P^1 - basis, i.e.: \ Phi(\underbrace{k}_{triangle}, \underbrace{i}_{number\ of\ Node}, \underbrace{j}_{1:a,2:b,3:c})$, where the basis function is defined $\theta_i(x,y) = a \cdot x + b \cdot y + c$.
- u_init is the start for the Newton algorithm and is set as identity.
- Size_W1 and Size_W2 are setting the size of the penilization area ω.
- M is the MassMatrix appearing in the boundary term penilization
- $(F)_i = \int_\Omega f \cdot \theta_i \ dx$, $u = \underline{\phi}^k = (\phi_1^k \dots \phi_n^k)^t$
- $A = DG(u), \ W_i = \int_\Omega \frac{\partial W}{\partial F}(\nabla \phi_S) : \nabla \theta_i$

Functions

- readmesh() reads the mesh from 'mesh.msh'
- setPhi() calculates the Phi(k,i,j)
- intM() calculates M
- intF() calculates F
- assMat(alpha,beta,gamma,delta,u) calculates A and W at every Newton step

2.3.2 assMat.m

assMat should be self-explaining. The values that are handed up are A and G.

2.3.3 calcPhi.m

calculates $\theta(X, Y)$ where X and Y are matrices in such a way, that θ needs to be avaluated for all X_{ij}, Y_{ij}. The values that are passed to calcPhi are

- k - # of triangle
- i - # of the basisfunction (1,2 or 3, as it is saved in ConnTable)
- bool - a boolean number, that is needed to change between $\binom{ax+by+c}{0}$ and $\binom{0}{ax+by+c}$

2.3.4 intF.m

one thing, that should be mentioned is what triquad is doing:
handing over the coordinates of the triangle and number of evaluation points, triquad hands back two matrices which contain the evaluation points and weights W_x, W_y which contain the weights.

2.3.5 intM.m

same thing as above (intF.m). Because one doesn't want to calculate the whole Mass Matrix, first it is checked if all triangle points are lying in the area ω. SizeW1 and SizeW2 are chosen in a way, that either the whole triangle lies in ω or the whole triangle lies outside.

2.3.6 write.m writeSol.m

these functions are used to save the solution data in a specific textfile

2.3.7 RB.m

this is the main function for the reduced basis method. It is very similar to FE.m. Differences:

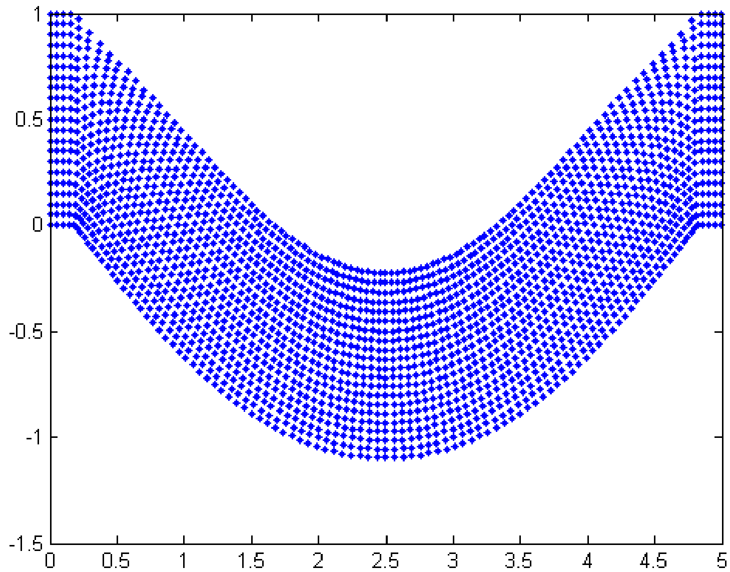
- the solution vectors need to be read, S=S(:,1:m) is used to reduce the numbers of basis vectors
- in the Newton iteration: instead of calling assMat(), we call assMatRB(), which is slightly different
- instead of calling intF(), we are calling intFSol(), since we now want to calculate $\int_\Omega f \cdot \delta u_r \; dx, \; \forall \delta u_r \in U_r$

Depending on the value for $\alpha, \beta, \gamma, \delta \; and \; f(x, y)$ we have different results:

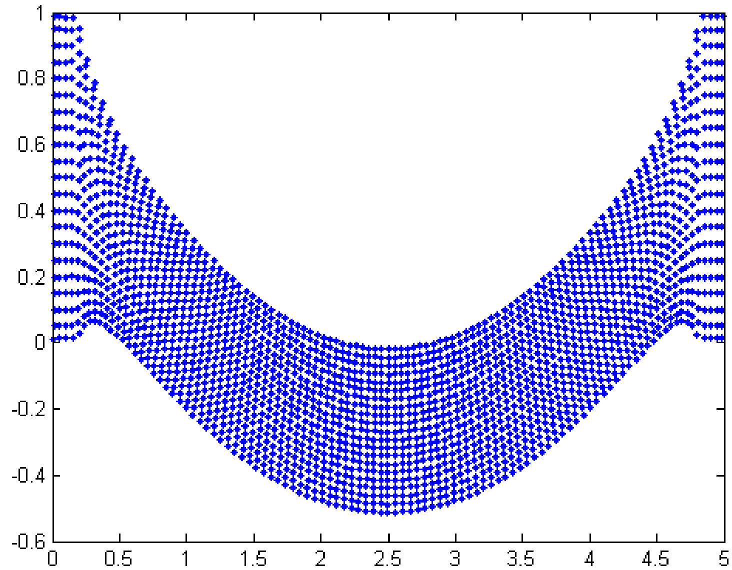
3 Plots

3.1 plots without reduced basis

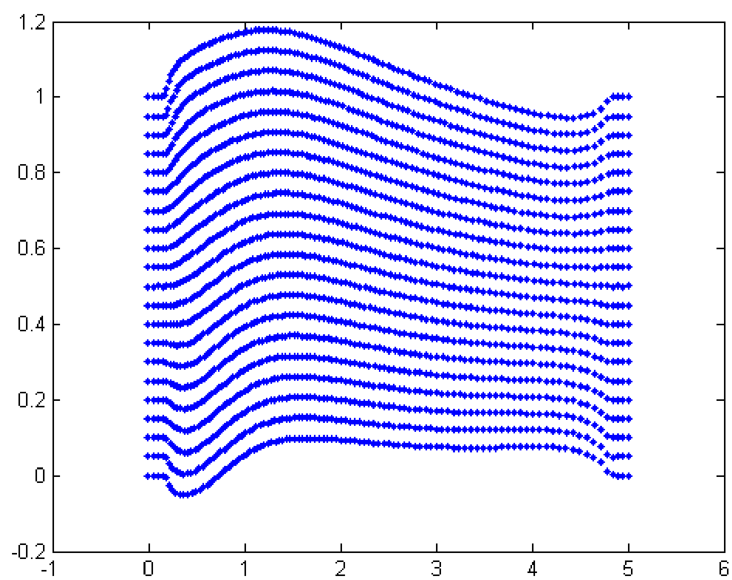
$f(x,y) = \begin{pmatrix} 0 \\ -1 \end{pmatrix},\ \alpha = \beta = \gamma = 1,\ \delta = 5$



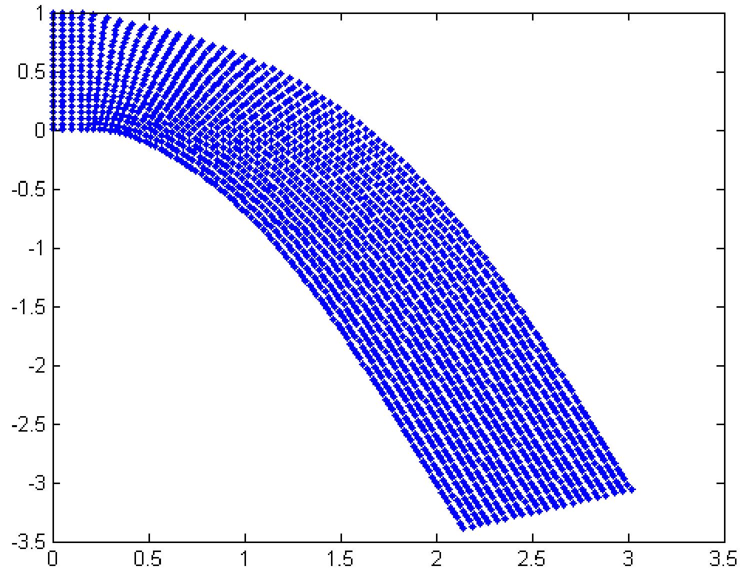
$$f(x,y) = \begin{pmatrix} 0 \\ -1 \end{pmatrix},\ \alpha = \beta = \gamma = 1,\ \delta = 1.5$$



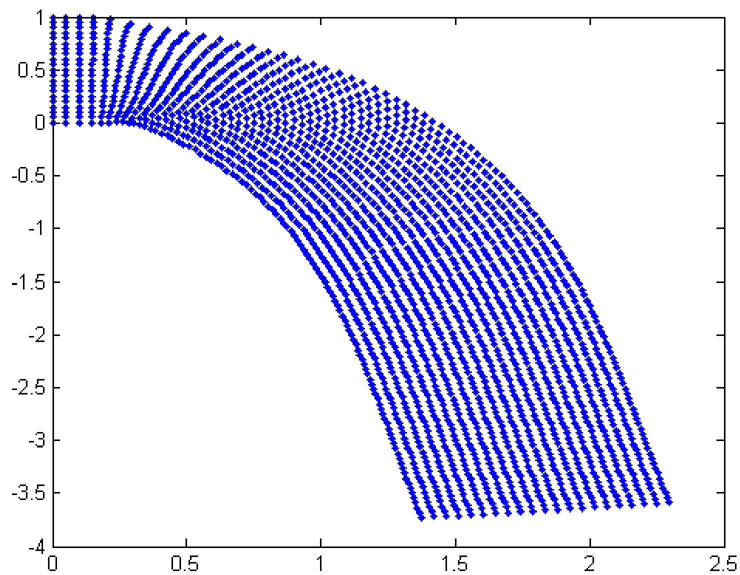
$$f(x,y) = \begin{pmatrix} -1.5 \\ 0 \end{pmatrix},\ \alpha = \beta = 0.7,\ \gamma = 1,\ \delta = 5$$



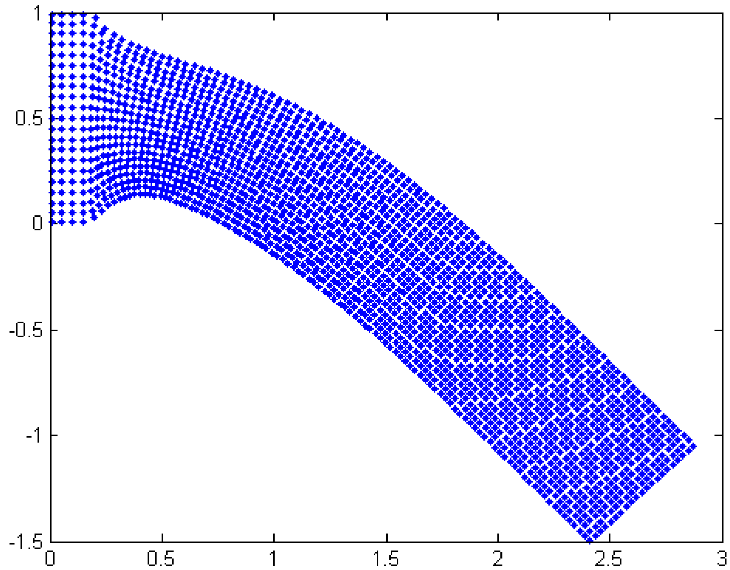
changing ω : $f(x,y) = \begin{pmatrix} 0 \\ -.1 \end{pmatrix}$, $\alpha = \beta = \gamma = 1$, $\delta = 5$



$f(x,y) = \begin{pmatrix} 0 \\ -.05x \end{pmatrix}$, $\alpha = \beta = \gamma = 1$, $\delta = 5$

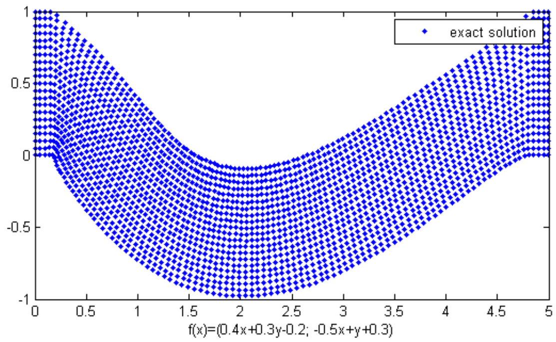


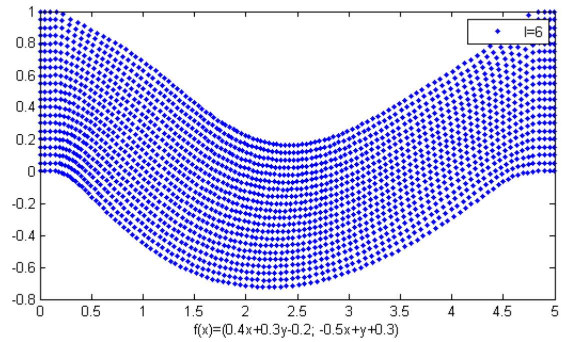
$$f(x,y) = \begin{pmatrix} 0 \\ -.03 \end{pmatrix},\ \alpha = \beta = \gamma = 1,\ \delta = 2$$

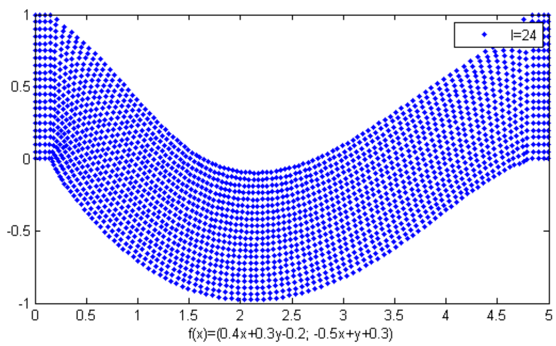


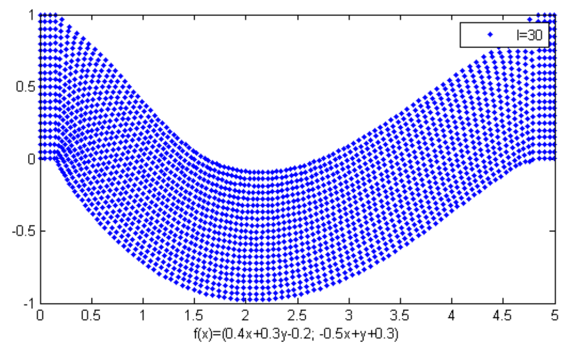
3.2 plots including reduced basis

$$f(x,y) = \begin{pmatrix} 0.4x + 0.3y - 0.2 \\ -0.5x + y + 0.3 \end{pmatrix},\ \alpha = \beta = \gamma = 1,\ \delta = 5$$

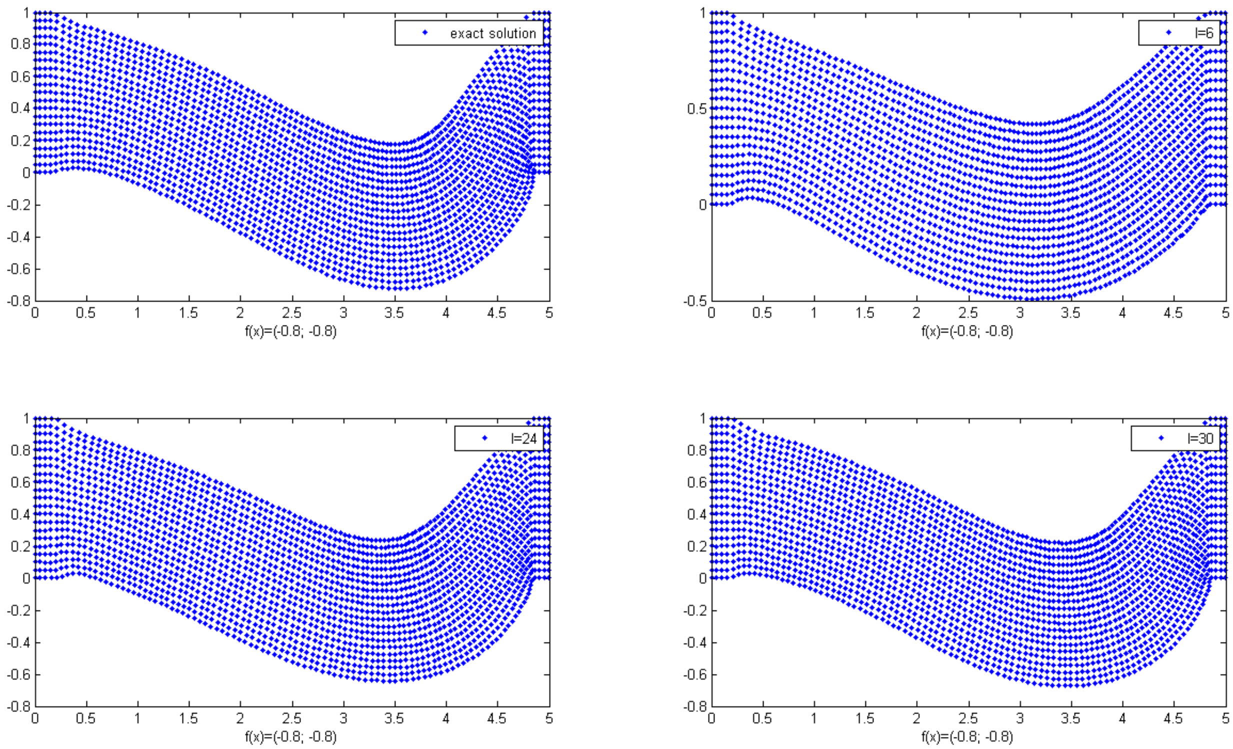




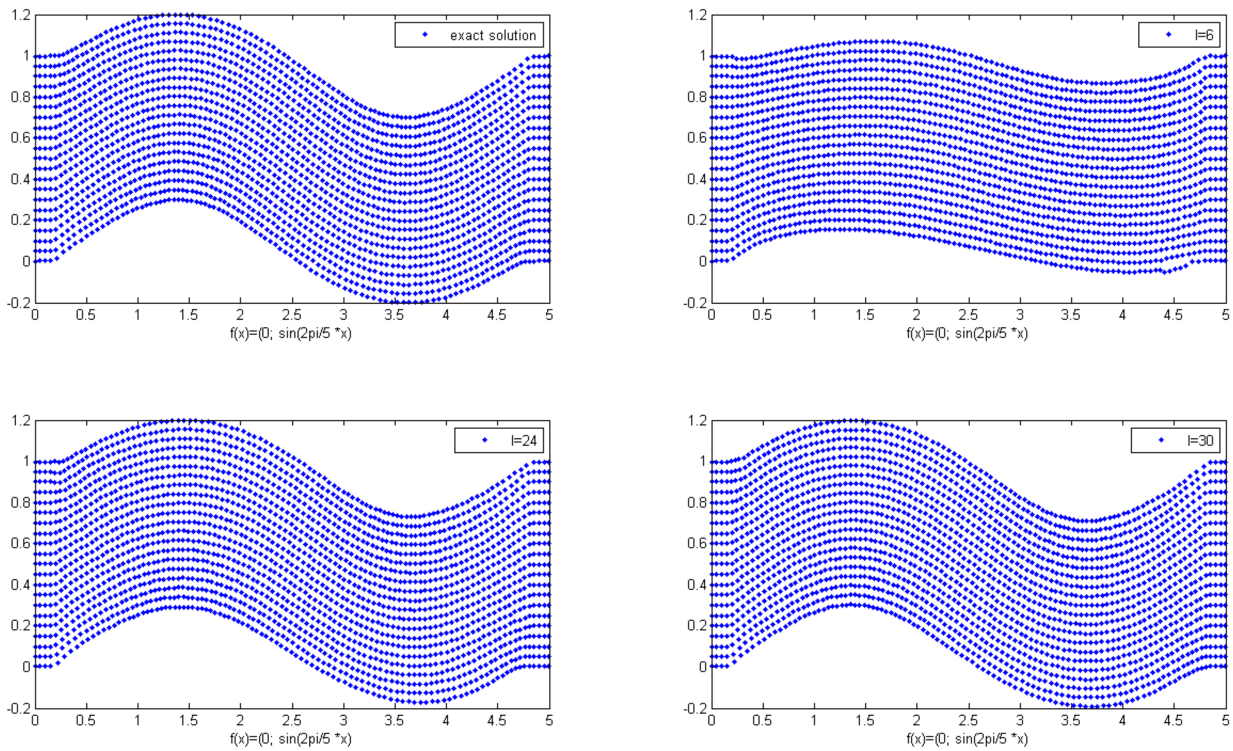




$$f(x,y) = \begin{pmatrix} -0.8 \\ -0.8 \end{pmatrix},\ \alpha = \beta = \gamma = 1,\ \delta = 5$$



$$f(x,y) = \begin{pmatrix} 0 \\ \sin\left(\frac{2\pi}{5} \cdot x\right) \end{pmatrix},\ \alpha = \beta = \gamma = 1,\ \delta = 5$$



4 Error table

$e = \underline{u}_{FE} - \underline{u}_{RB}$

$f_1(x,y) = \begin{pmatrix} -0.8 \\ -0.8 \end{pmatrix}$	$\|e\|_2$	$\|e\|_\infty$
I=6	9.04025	0.33289
I=12	2.61462	0.12019
I=18	1.5115	0.07257
I=24	0.72561	0.02598
I=30	0.61432	0.02661

$f_2(x,y) = \begin{pmatrix} 0.4x + 0.3y - 0.2 \\ -0.5x + y + 0.3 \end{pmatrix}$	$\|e\|_2$	$\|e\|_\infty$
I=6	7.23828	0.29562
I=12	4.47455	0.18567
I=18	3.52092	0.14728
I=24	2.22963	0.0936
I=30	1.29669	0.05495

$f_3(x,y) = \begin{pmatrix} 0 \\ \sin\left(\frac{2\pi}{5} \cdot x\right) \end{pmatrix}$	$\|e\|_2$	$\|e\|_\infty$
I=6	4.93955	0.17723
I=12	3.39434	0.1344
I=18	2.01984	0.08634
I=24	1.24071	0.05417
I=30	0.44359	0.01943

References

[1] Phillippe G. Ciarlet *Mathematical Elasticity, Volume 1: Three Dimensional Elasticity.* North-Holland, 1988.

[2] Patrice Hauret. Reduced Basis Approach for Nonlinear Elasticity. *February 17, 2009*