Reduced basis approach for nonlinear Elasticity

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Outline

1. Theory
   - theory
2. Numerics
   - numerics
3. Remarks
   - remarks

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Equations of nonlinear Elasticity

We will look at the equations:

- $$\int_\Omega \frac{\partial W}{\partial F}(\nabla\phi) : \nabla\theta_i \; dx = \int_\Omega f \cdot \theta_i \; dx$$
  in the compressible case and
- $$\begin{cases} \int_\Omega \frac{\partial W}{\partial F}(\nabla\phi) : \nabla\theta_i \; dx + \int_\Omega p \; cof \nabla\phi : \nabla\theta_i \; dx & = \int_\Omega f \cdot \theta_i \; dx \\ \int_\Omega p_i(\det\nabla\phi - 1) & = 0 \end{cases}$$
  in the incompressible case
- with
  $W(F) = \alpha \cdot \|F\|^2 + \beta \cdot \|cof \; F\|^2 + \gamma \cdot \|\det F\|^2 - \delta \cdot \ln(\det F)$
  These equations must be fulfilled for all testfunctions $\theta_i \in \Theta_s$, where Θ_s denotes the finite element solutions subspace.

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Basic notions

- Ω is a bounded, open, connected subset of $\mathbb{R}^3$ with suffiencently smooth boundary. $\bar{\Omega}$ represents the volume occupied by a body before it is deformed and is called **reference configuration**.
- One may write

$$\phi = \mathbf{id} + \mathbf{u}$$

with

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homogeneous hyperelastic material

Definition

A homogeneous elastic material with response function

$$\hat{\mathbf{T}} : \mathbb{M}^3_+ \to \mathbb{M}^3$$

is called **homogeneous hyperelastic** if there exists a function

$$\hat{W} : \mathbb{M}^3_+ \to \mathbb{R}$$

differentiable with respect to the variable $\mathbf{F} \in \mathbb{M}^3_+$, such that

$$\hat{\mathbf{T}}(\mathbf{F}) = \frac{\partial \hat{W}}{\partial \mathbf{F}}(\mathbf{F}), \qquad \forall \mathbf{F} \in \mathbb{M}^3_+$$

. The function $\hat{W}$ is called **stored energy function**.

minimal property

Fact

The equations of equilibrium are formally equivalent to the equations

$$I'(\phi)\theta = 0$$

with

$$I(\psi) = \int_\Omega \hat{W}(\nabla\psi(x))dx - \{F(\psi) + G(\psi)\}.$$

So we are looking of a *minimum* of the functional I!

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Finite element system

We were solving the system:

$$\int_\Omega \frac{\partial W}{\partial F}(\nabla \phi_S) : \nabla \theta_i \; dx + \frac{1}{\varepsilon} M \cdot \underline{\phi} = \int_\Omega f \cdot \theta_i \; \forall i \in \{1 \dots n\}$$

$$\phi_S(x,y) = \sum_{i=1}^{n} \phi_i \theta_i(x,y) \qquad \underline{\phi} = (\phi_1 \dots \phi_n)^t$$

for $\theta_i \in P^1$ and

$$\theta_i(x_j) = \begin{cases} \delta_{ij} & for \;\; i = 1 \dots m \\ \delta_{(i-m)j} & for \;\; i = m+1 \dots n \end{cases}, \qquad M_{ij} = \int_\omega \theta_i \cdot \theta_j \; dx$$

System nonlinear $\Rightarrow$ Finite Elements + Newton Algorithm for the ϕ_i in ϕ_S

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Newton Algorithm

- $$\begin{cases} DG(\underline{\phi}^k)\triangle\phi & = G(\underline{\phi}^k) \\ \underline{\phi}^{k+1} & = \underline{\phi}^k - \triangle\phi \end{cases} \qquad \underline{\phi}^k = (\phi_1^k \dots \phi_n^k)^t$$

- $$\left[DG(\underline{\phi}^k)\right]_{ij} = \int_\Omega \frac{\partial}{\partial \phi_j}\left(\frac{\partial W}{\partial F}(\nabla\phi)\right) : \nabla\theta_i \, dx + \frac{1}{\varepsilon}\cdot M$$

- $$\left[G(\underline{\phi}^k)\right]_i = \int_\Omega \frac{\partial W}{\partial F}(\nabla\phi_S) : \nabla\theta_i \, dx + \frac{1}{\varepsilon} M \cdot \underline{\phi} - \int_\Omega f\cdot\theta \, dx$$

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Reduced basis approach

basic Idea

The basic idea is to precompute solutions and use those solutions as basis functions in our finite elements solution subspace i.e. $\Theta_S = U_r = span\{u_i, i = 1 \dots l\}$, where u_i are the displacements of the solutions ϕ_S.

We end up solving

$$\int_\Omega \frac{\partial W}{\partial F}(Id + \nabla u_r) : \nabla \delta u_r \, dx = \int_\Omega f \cdot \delta u_r \, dx, \; \forall \delta u_r \in U_r$$

$\Longrightarrow$ For the reduced basis system the Matrix in the Newton system is not sparse anymore!

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Plots

$\Longrightarrow$ go to Matlab!

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Numerical Results:

We have chosen $f(x,y)$ linear in both components,
$\alpha = \beta = \gamma = 1,\ \delta = 5$, Number Nodes = 2121
Domain= $[0,5] \times [0,1]$, $e = \underline{u}_{FE} - \underline{u}_{RB}$

$f_1(x,y) = \begin{pmatrix} -0.8 \\ -0.8 \end{pmatrix}$	$\|e\|_2$	$\|e\|_\infty$
l=6	9.04025	0.33289
l=12	2.61462	0.12019
l=18	1.5115	0.07257
l=24	0.72561	0.02598
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Numerical Results 2:

$f_2(x,y) = \begin{pmatrix} 0.4x + 0.3y - 0.2 \\ -0.5x + y + 0.3 \end{pmatrix}$	$\|e\|_2$	$\|e\|_\infty$
l=6	7.23828	0.29562
l=12	4.47455	0.18567
l=18	3.52092	0.14728
l=24	2.22963	0.0936
l=30	1.29669	0.05495

$f_3(x,y) = \begin{pmatrix} 0 \\ \sin\left(\frac{2\pi}{5} \cdot x\right) \end{pmatrix}$	$\|e\|_2$	$\|e\|_\infty$
l=6	4.93955	0.17723
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Times

We had converging problems for $l = 24 \wedge l = 30$ due to the basis functions, that are added!

	f_1	f_2	f_3
exact	37.61s	45.15s	29.27s
l=6	6.8s	6.83s	5.27s
l=12	24.15s	23.83s	22.24s
l=18	61.04s	59.92s	49.33s
l=24	354.96s	345.11s	101.32s
l=30	566.57s	506.87s	498.63s

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Remarks

- In my code the FE-Algorithm was almost as fast as the RB-Algorithm with l=12. The RB-Algorithm can be speed up, but the aim should be to keep the RB-basis low dimensional.
- It is very important **how** the basis functions in the RB-Algorithm are chosen! In my case the last added solutions are badly chosen, because the Jacobian in the Newton-Algorithm becomes badly scaled.
- It seems like even RHSs that are not linear are still approximated in a good way, so even if one has computed the basisfunctions in a certain set, one can extend the possible RHSs to a bigger space.

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